POLITECNICO DI TORINO

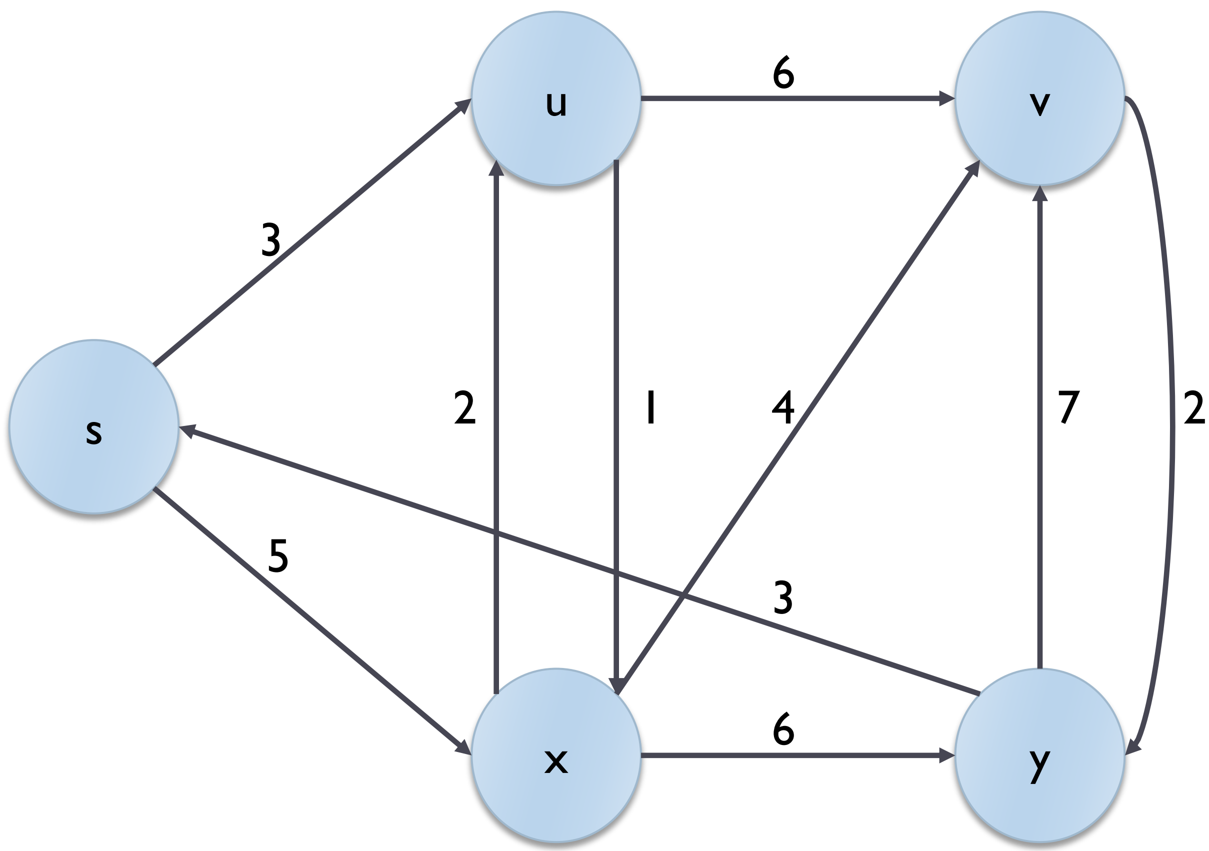
e-Lite

Graphs: Finding shortest paths

Tecniche di Programmazione – A.A. 2017/2018

Example

What is the shortest path between s and v ?



Summary

- Definitions
- Floyd-Warshall algorithm
- Bellman-Ford-Moore algorithm
- Dijkstra algorithm

Definitions

Graphs: Finding shortest paths

Definition: weight of a path

- Consider a directed, weighted graph G=(V,E), with weight function w: E→***R***
  - This is the general case: undirected or un-weighted are automatically included
- The weight w(p) of a path p is the sum of the weights of the edges composing the path

$$w(p) = \sum_{(u,v) \in p} w(u, v)$$

Definition: shortest path

- The shortest path between vertex u and vertex v is defined as the mininum-weight path between u and v, if the path exists.
- The weight of the shortest path is represented as $\delta(u,v)$
- If v is not reachable from u, then $\delta(u,v)=\infty$

Finding shortest paths

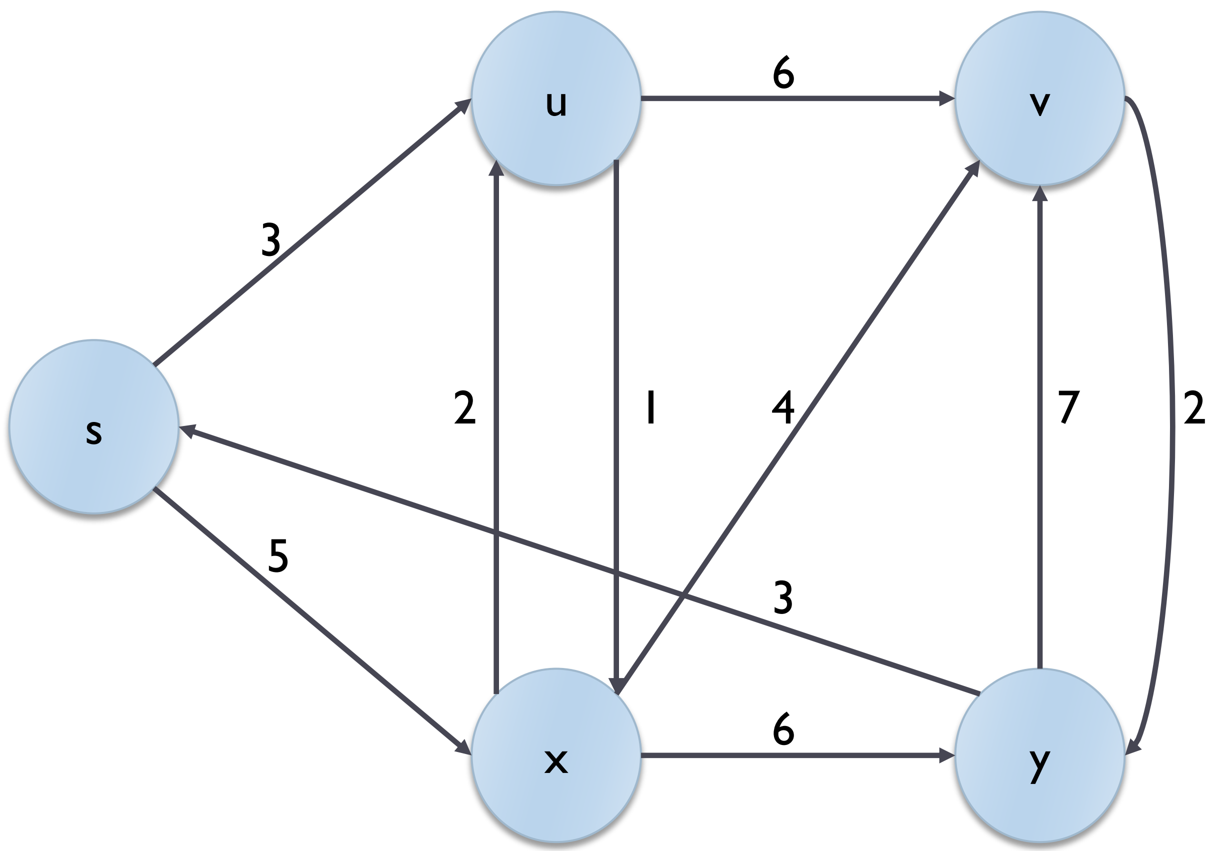
- Single-source shortest path (SS-SP)
  - Given u and v, find the shortest path between u and v
  - Given u, find the shortest path between u and any other vertex
- All-pairs shortest path (AP-SP)
  - Given a graph, find the shortest path between any pair of vertices

What to find?

- Depending on the problem, you might want:
  - The **value** of the shortest path weight
    - Just a real number
  - The **actual path** having such minimum weight
    - For simple graphs, a sequence of vertices. For multigraphs, a sequence of edges

Example

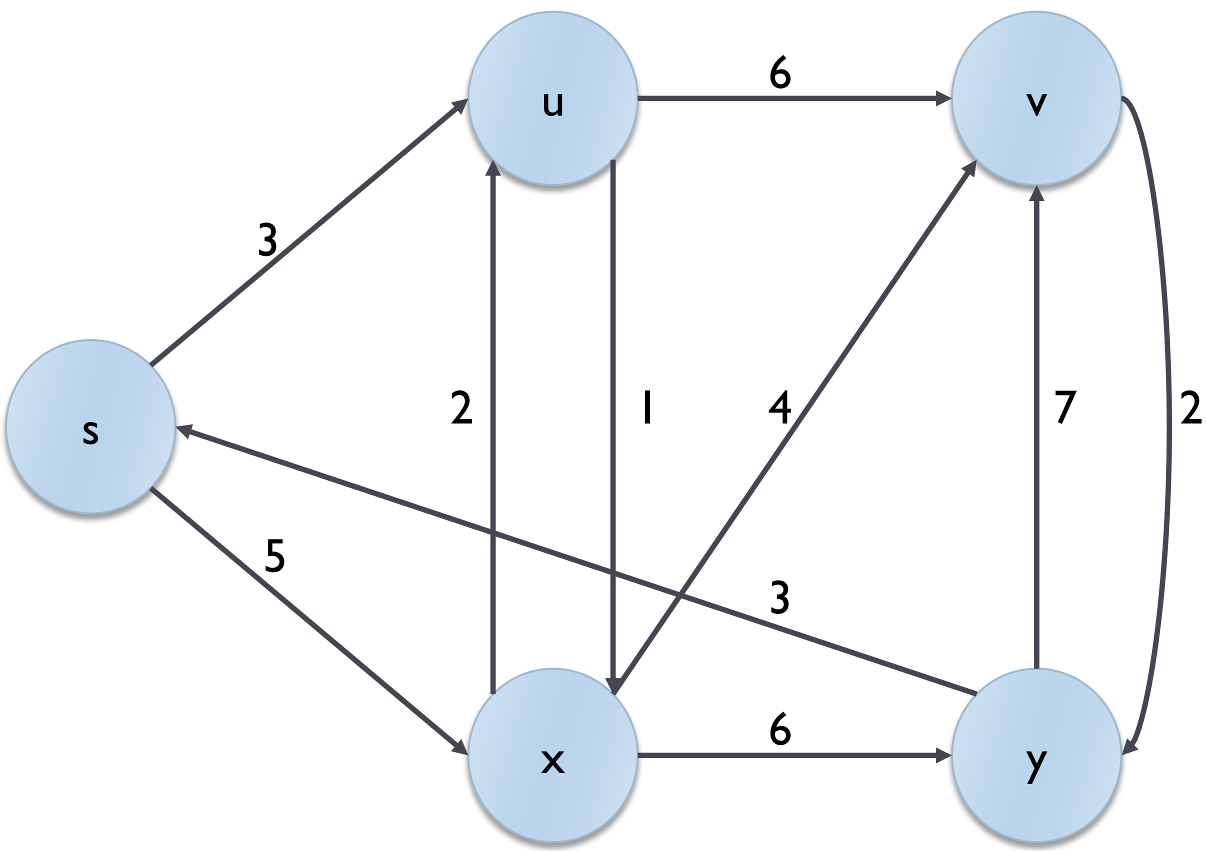
What is the shortest path between s and v ?



Representing shortest paths

- To store all shortest paths from a single source u, we may add
  - For each vertex v, the **weight** of the shortest path $\delta(u,v)$
  - For each vertex v, the "**preceding**" vertex $\pi(v)$ that allows to reach v in the shortest path
    - For multigraphs, we need the preceding edge
- Example:
  - Source vertex: u
  - For any vertex v:
    - `double v.weight ;`
    - `Vertex v.preceding ;`

Example



π

Vertex	Previous
s	NULL
u	s
x	u
v	x
y	v

δ

Vertex	Weight
s	0
u	3
x	4
v	8
y	10

Lemma

- The "previous" vertex in an intermediate node of a minimum path does **not** depend on the **final** destination
- Example:
  - Let p_1 = shortest path between u and v_1
  - Let p_2 = shortest path between u and v_2
  - Consider a vertex $w \in p_1 \cap p_2$
  - The value of $\pi(w)$ may be chosen in a single way and still guarantee that both p_1 and p_2 are shortest

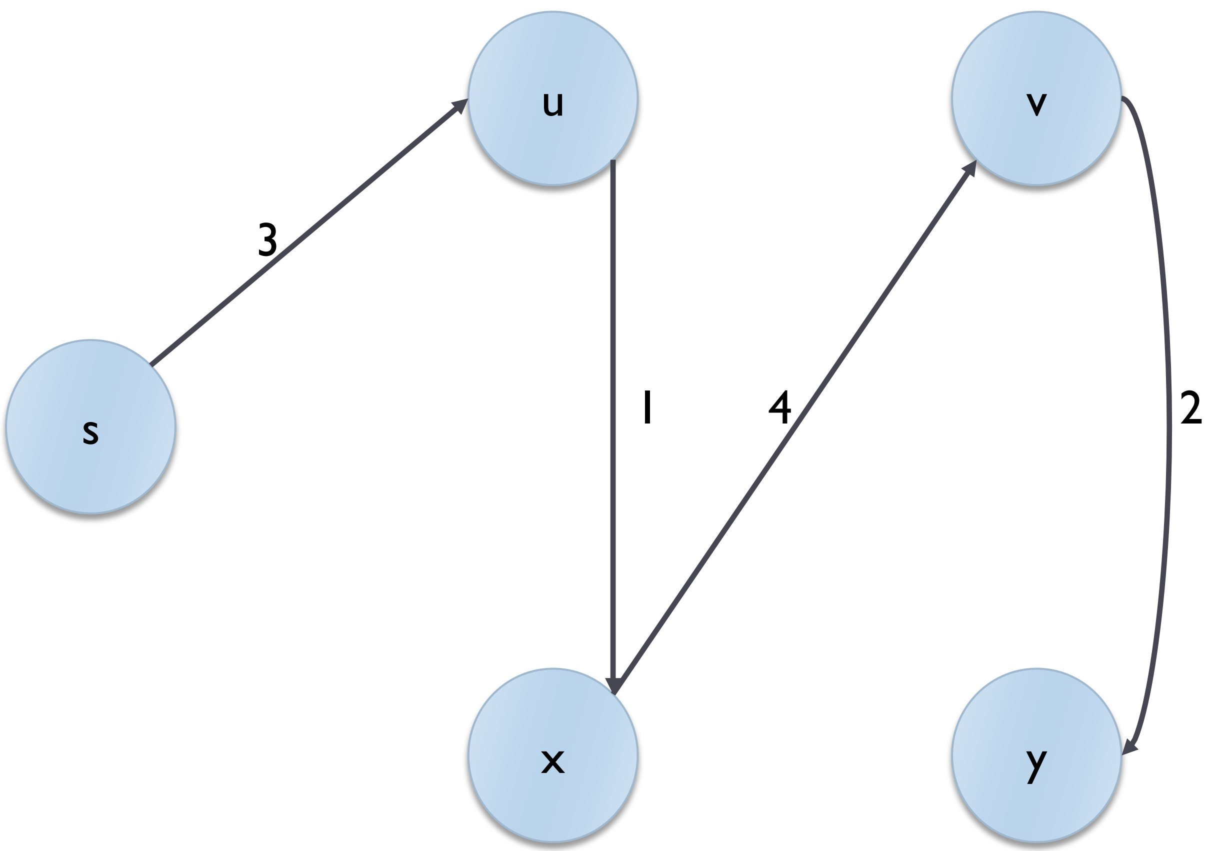
Shortest path graph

- Consider a source node u
- Compute all shortest paths from u
- Consider the relation $E\pi$ = { (v.preceding, v) }
- $E\pi \subseteq E$
- $V\pi$ = { v $\in$ V : v reachable from u }
- $G\pi = G(V\pi, E\pi)$ is a subgraph of G(V,E)
- $G\pi$: the predecessor-subgraph

Shortest path tree

- $G\pi$ is a tree (due to the Lemma) rooted in u
- In $G\pi$, the (unique) paths starting from u are always shortest paths
- $G\pi$ is not unique, but all possible $G\pi$ are equivalent (same weight for every shortest path)

Example



π

Vertex	Previous
s	NULL
u	s
x	u
v	x
y	v

δ

Vertex	Weight
s	0
u	3
x	4
v	8
y	10

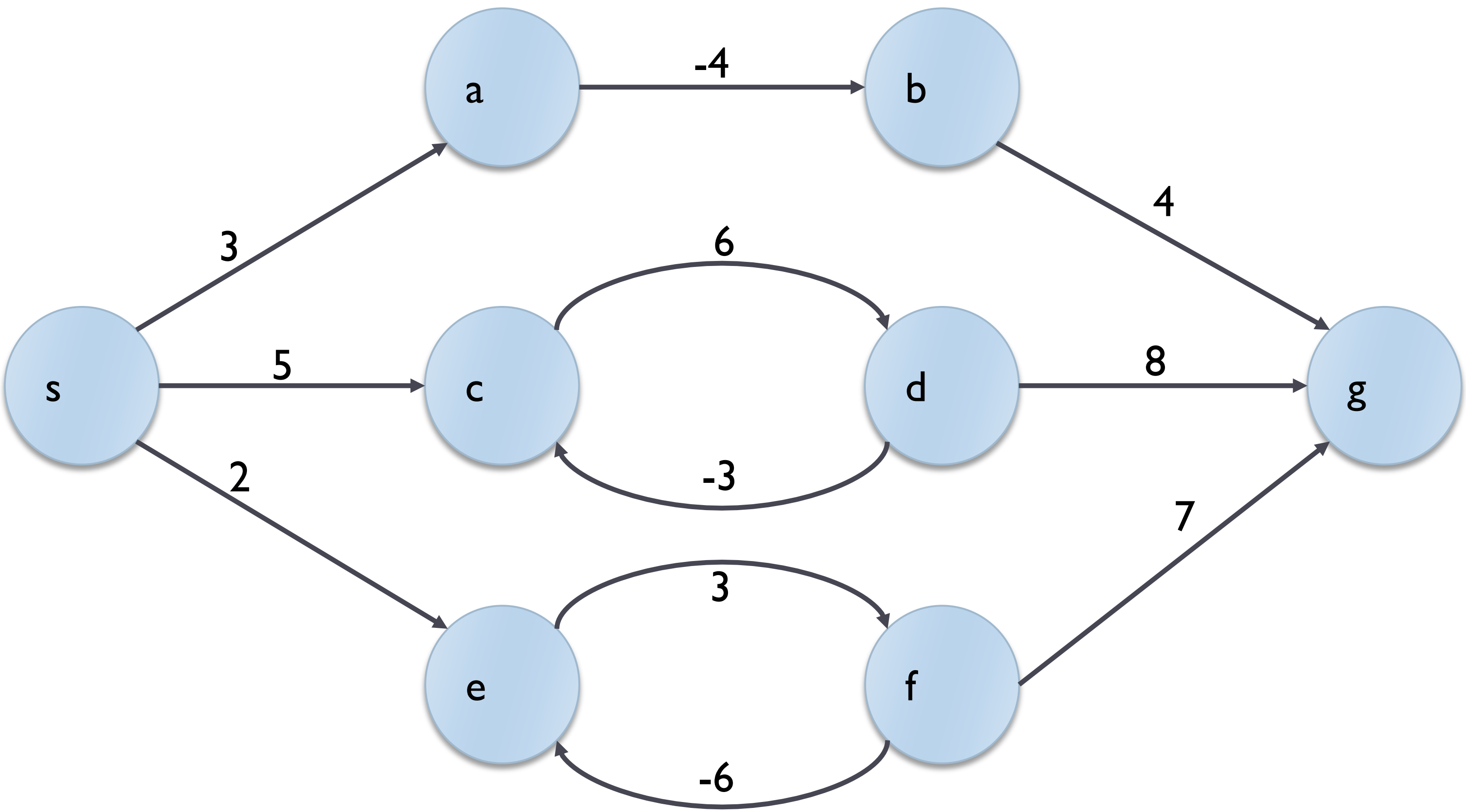
Special case

- If G is an un-weighted graph, then the shortest paths may be computed just with a breadth-first visit

Negative-weight cycles

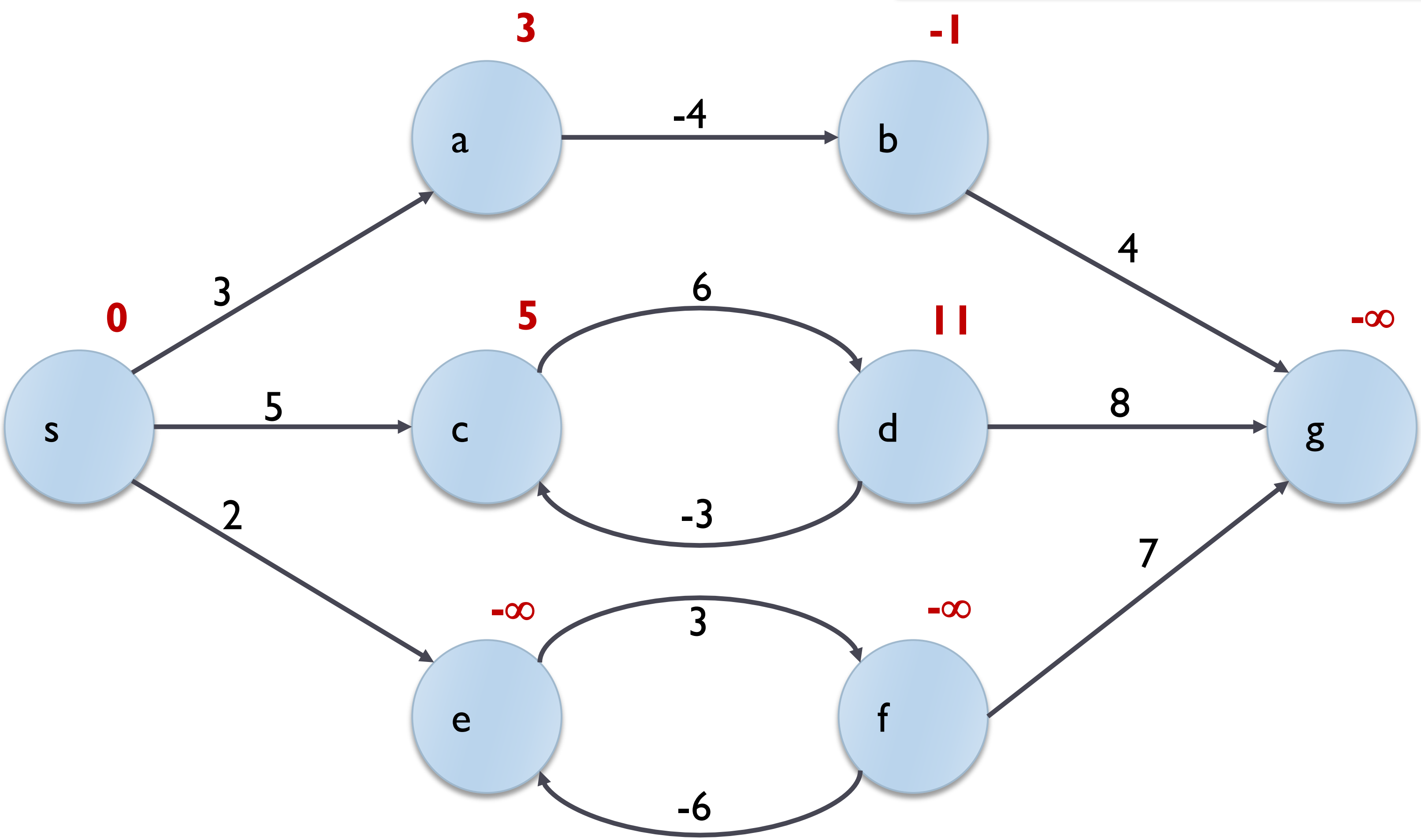
- Minimum paths cannot be defined if there are negative-weight cycles in the graph
- In this case, the minimum path does not exist, because you may always decrease the path weight by going once more through the loop.
- Conventionally, in these case we say that the path weight is $-\infty$.

Example



Example

Minimum-weight paths from source vertex s



Lemma

- Consider an ordered weighted graph G=(V,E), with weight function w: E→***R***.
- Let p=<v_1, v_2, …, v_k> a shortest path from vertex v_1 to vertex v_k.
- For all i,j such that $1 \leq i \leq j \leq k$, let p_{ij}=<v_i, v_{i+1}, …, v_j> be the sub-path of p, from vertex v_i to vertex v_j.

- Therefore, p_{ij} is a shortest path from v_i to v_j.

Corollary

- Let p be a shortest path from s to v
- Consider the vertex u, such that (u,v) is the last edge in the shortest path
- We may decompose p (from s to v) into:
  - A sub-path from s to u
  - The final edge (u,v)
- Therefore
  - $\delta(s,v)=\delta(s,u)+w(u,v)$

Lemma

- If we arbitrarily chose the vertex u', then for all edges (u',v)$\in$E we may say that
  - $\delta(s,v) \leq \delta(s,u') + w(u',v)$

Relaxation

- Most shortest-path algorithms are based on the relaxation technique
- It consists of
  - Vector d[u] represents $\delta(s,u)$
  - Keeping track of an updated estimate d[u] of the shortest path towards each node u
  - Relaxing (i.e., updating) d[v] (and therefore the predecessor π[v]) whenever we discover that node v is more conveniently reached by traversing edge (u,v)

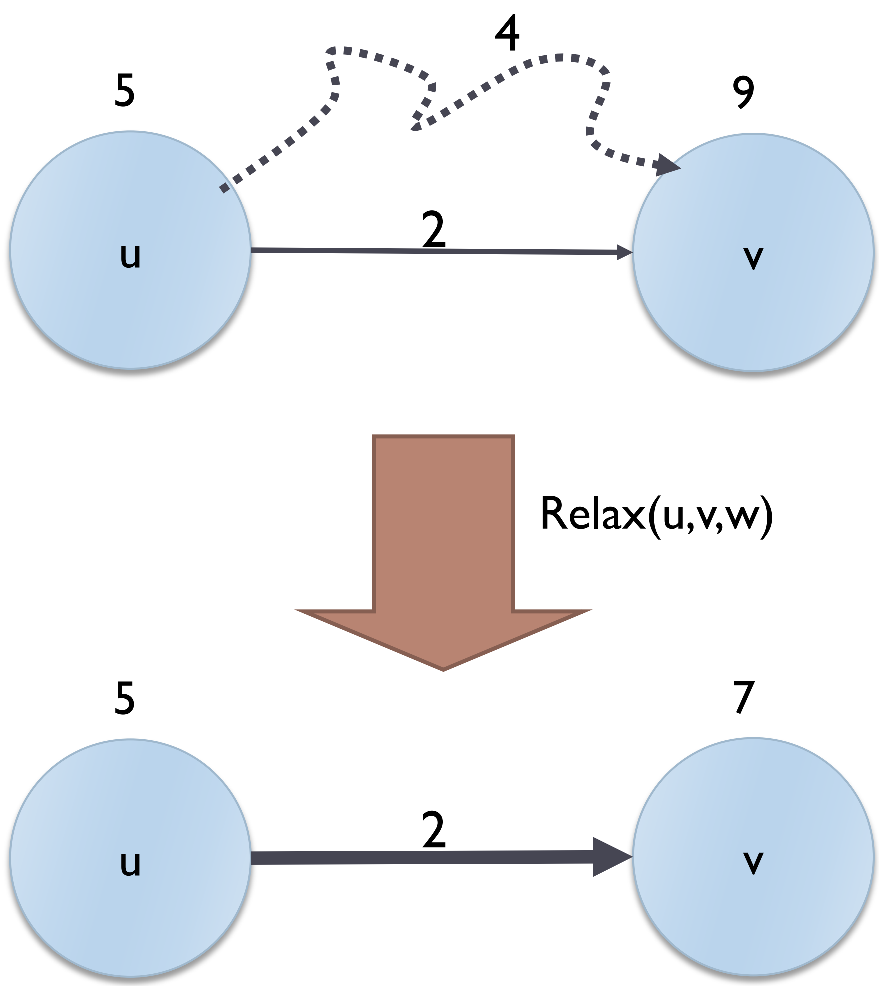
Initial state

- Initialize-Single-Source(G(V,E), s)
  1. **for** all vertices v $\in$ V
  2. **do**
     1. d[v]$\leftarrow\infty$
     2. π[v]$\leftarrow$NIL
  3. d[s]$\leftarrow$0

Relaxation

- We consider an edge (u,v) with weight w
- Relax(u, v, w)
  1. **if** d[v] > d[u]+w(u,v)
  2. **then**
     1. d[v] ← d[u]+w(u,v)
     2. π[v] ← u

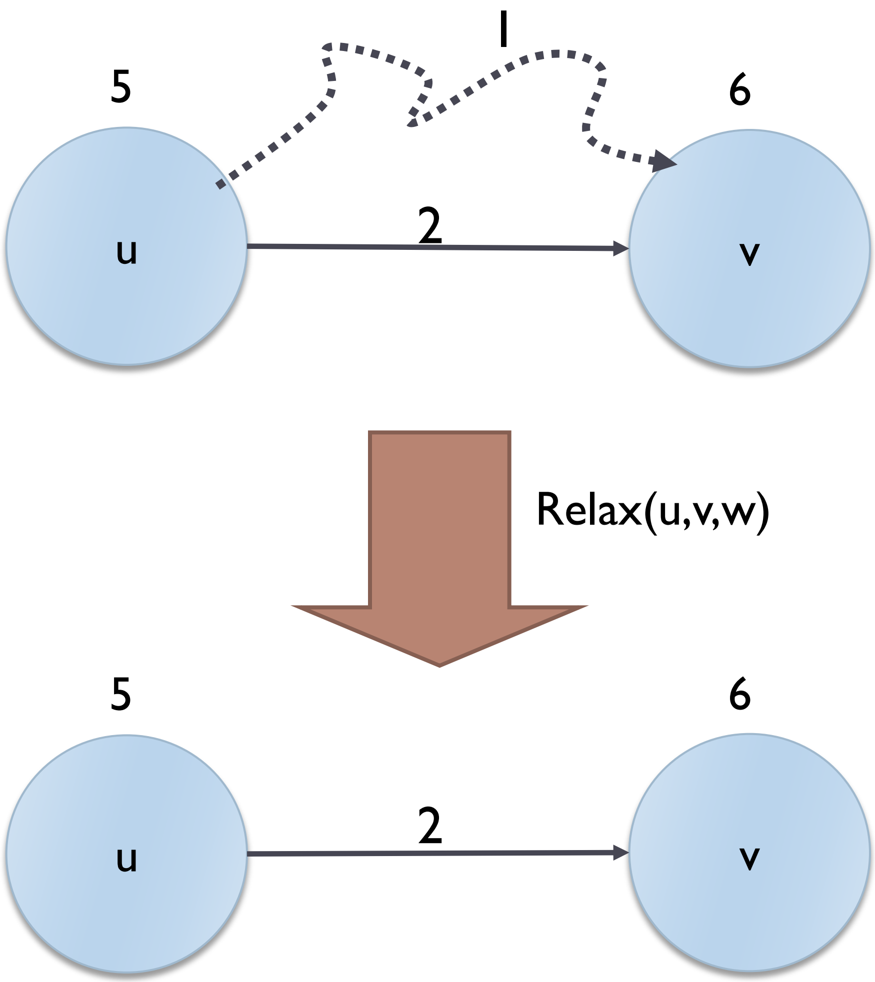
Example 1



Before:
Shortest known path to v weights 9, does not contain (u,v)

After:
Shortest path to v weights 7, the path includes (u,v)

Example 2



Before:
Shortest path to v weights 6, does not contain (u,v)

After:
No relaxation possible, shortest path unchanged

Lemma

- Consider an ordered weighted graph G=(V,E), with weight function w: $E \rightarrow \boldsymbol{R}$.
- Let (u,v) be an edge in G.
- After relaxation of (u,v) we may write that:
  - $d[v] \leq d[u]+w(u,v)$

Lemma

- Consider an ordered weighted graph G=(V,E), with weight function w: $E \rightarrow \boldsymbol{R}$ and source vertex $s \in V$. Assume that G has no negative-weight cycles reachable from s.

- Therefore
  - After calling Initialize-Single-Source(G,s), the predecessor subgraph $G\pi$ is a rooted tree, with s as the root.
  - Any relaxation we may apply to the graph does not invalidate this property.

Lemma

- Given the previous definitions.
- Apply any possible sequence of relaxation operations
- Therefore, for each vertex v
  - $d[v] \geq \delta(s,v)$
- Additionally, if $d[v] = \delta(s,v)$, then the value of d[v] will not change anymore due to relaxation operations.

Shortest path algorithms

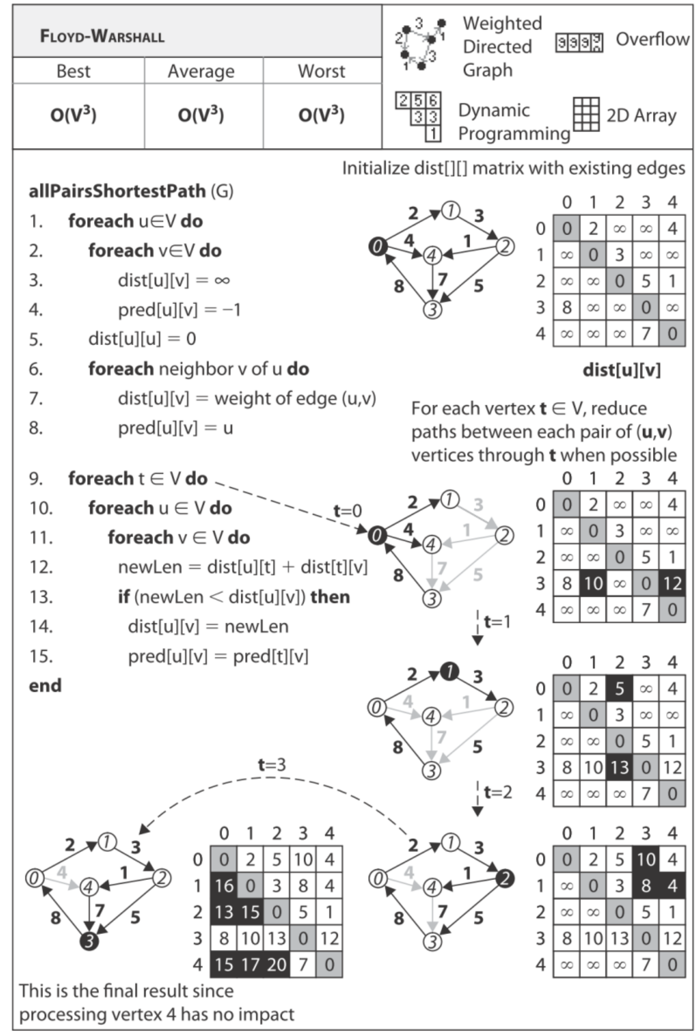
- Various algorithms
- Differ according to one-source or all-sources requirement
- Adopt repeated relaxation operations
- Vary in the order of relaxation operations they perform
- May be applicable (or not) to graph with negative edges (but no negative cycles)

Floyd-Warshall algorithm

Graphs: Finding shortest paths

Floyd-Warshall algorithm

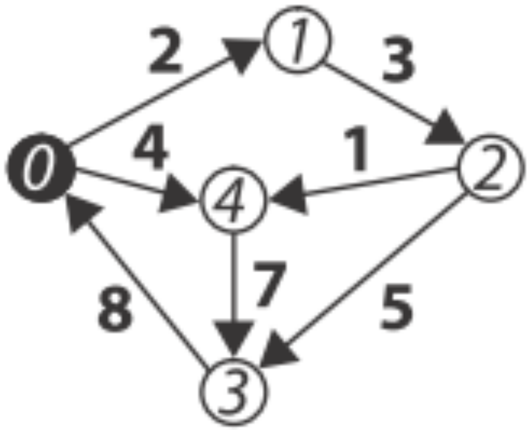
- Computes the all-source shortest path (AP-SP)
- dist[i][j] is an n-by-n matrix that contains the length of a shortest path from vi to vj.
- if dist[u][v] is ∞, there is no path from u to v
- pred[s][j] is used to reconstruct an actual shortest path: stores the predecessor vertex for reaching vj starting from source vs



Floyd-Warshall: initialization

```
allPairsShortestPath (G)
    foreach u∈V do
        foreach v∈V do
            dist[u][v] = ∞
            pred[u][v] = −1
        dist[u][u] = 0
        foreach neighbor v of u do
            dist[u][v] = weight of edge (u,v)
            pred[u][v] = u
```

Example, after initialization



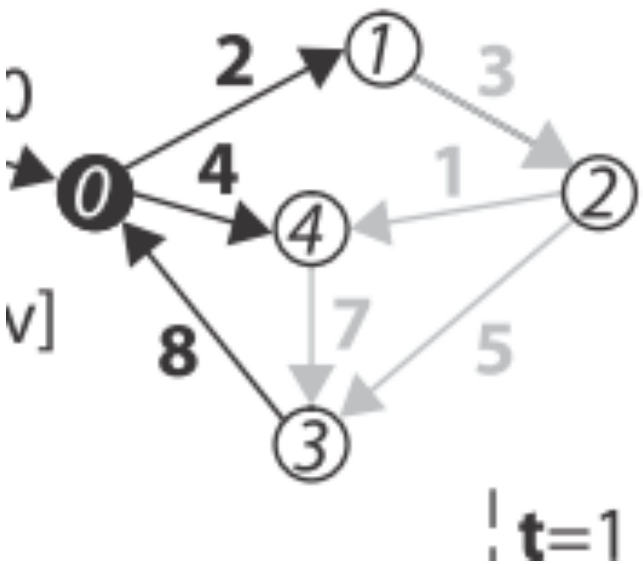
	0	1	2	3	4
0	0	2	∞	∞	4
1	∞	0	3	∞	∞
2	∞	∞	0	5	1
3	8	∞	∞	0	∞
4	∞	∞	∞	7	0

dist[u][v]

Floyd-Warshall: relaxation

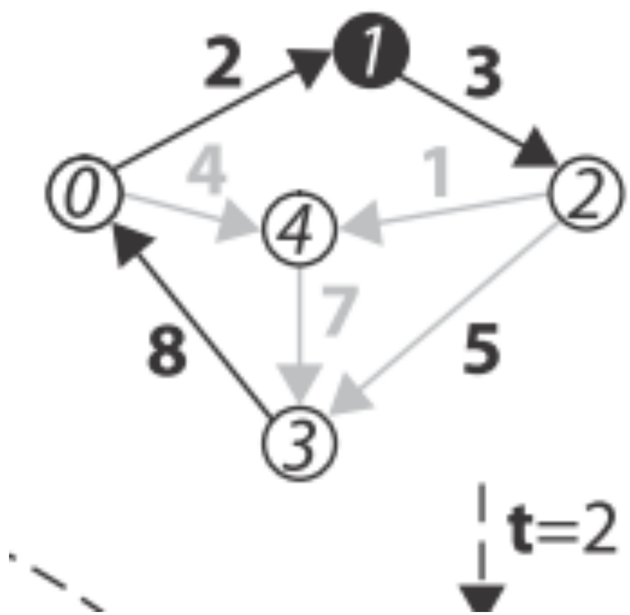
```
foreach t ∈ V do
    foreach u ∈ V do
        foreach v ∈ V do
            newLen = dist[u][t] + dist[t][v]
            if (newLen < dist[u][v]) then
                dist[u][v] = newLen
                pred[u][v] = pred[t][v]
```

Example, after step t=0



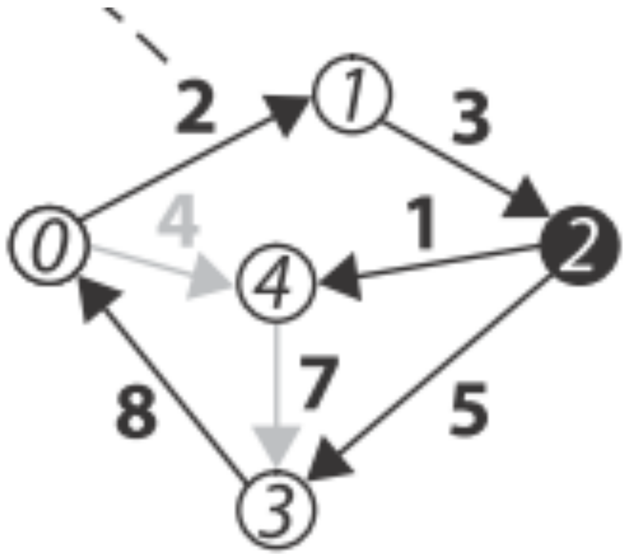
	0	1	2	3	4
0	0	2	∞	∞	4
1	∞	0	3	∞	∞
2	∞	∞	0	5	1
3	8	10	∞	0	12
4	∞	∞	∞	7	0

Example, after step t=1



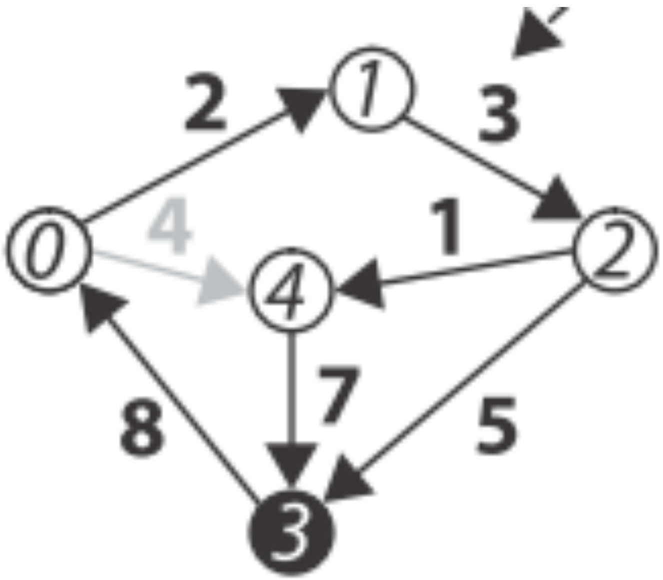
	0	1	2	3	4
0	0	2	5	∞	4
1	∞	0	3	∞	∞
2	∞	∞	0	5	1
3	8	10	13	0	12
4	∞	∞	∞	7	0

Example, after step t=2



	0	1	2	3	4
0	0	2	5	10	4
1	∞	0	3	8	4
2	∞	∞	0	5	1
3	8	10	13	0	12
4	∞	∞	∞	7	0

Example, after step t=3



	0	1	2	3	4
0	0	2	5	10	4
1	16	0	3	8	4
2	13	15	0	5	1
3	8	10	13	0	12
4	15	17	20	7	0

Complexity

- The Floyd-Warshall is basically executing 3 nested loops, each iterating over all vertices in the graph
- Complexity: $O(V^3)$

Implementation

org.jgrapht.alg

Class FloydWarshallShortestPaths<V,E>

```
java.lang.Object
  └ org.jgrapht.alg.FloydWarshallShortestPaths<V,E>
```

```
public class FloydWarshallShortestPaths<V,E>
extends java.lang.Object
```

The Floyd-Warshall algorithm finds all shortest paths (all n^2 of them) in O(n^3) time. This also works out the graph diameter during the process.

Author:

Tom Larkworthy, Soren Davidsen

Constructor Summary
FloydWarshallShortestPaths(Graph<V,E> graph)

Method Summary	
double	getDiameter()
Graph<V,E>	getGraph()
GraphPath<V,E>	getShortestPath(V a, V b) Get the shortest path between two vertices.
java.util.List<GraphPath<V,E>>	getShortestPaths(V v) Get shortest paths from a vertex to all other vertices in the graph.
int	getShortestPathsCount()
double	shortestDistance(V a, V b) Get the length of a shortest path.

Bellman-Ford-Moore Algorithm

Graphs: Finding shortest paths

Bellman-Ford-Moore Algorithm

- Solution to the single-source shortest path (SS-SP) problem in graph theory
- Based on relaxation (for every vertex, relax all possible edges)
- Does not work in presence of negative cycles
  - but it is able to detect the problem
- $O(V \cdot E)$

Bellman-Ford-Moore Algorithm

```
dist[s] ← 0                     (distance to source vertex is zero)
for  all v ∈ V–{s}
     do  dist[v] ←∞             (set all other distances to infinity)
for i ← 0 to |V|
     for all (u, v) ∈ E
         do  if  dist[v] > dist[u] + w(u, v)        (if new shortest path found)
                 then  d[v] ←d[u] + w(u, v)         (set new value of shortest path)
                       (if desired, add traceback code)

for all (u, v) ∈ E              (sanity check)
         do  if  dist[v] > dist[u] + w(u, v)
                 then  PANIC!
```

Dijkstra's Algorithm

Graphs: Finding shortest paths

Dijkstra's algorithm

- Solution to the single-source shortest path (SS-SP) problem in graph theory
- Works on both directed and undirected graphs
- All edges must have nonnegative weights
  - the algorithm would miserably fail
- Greedy

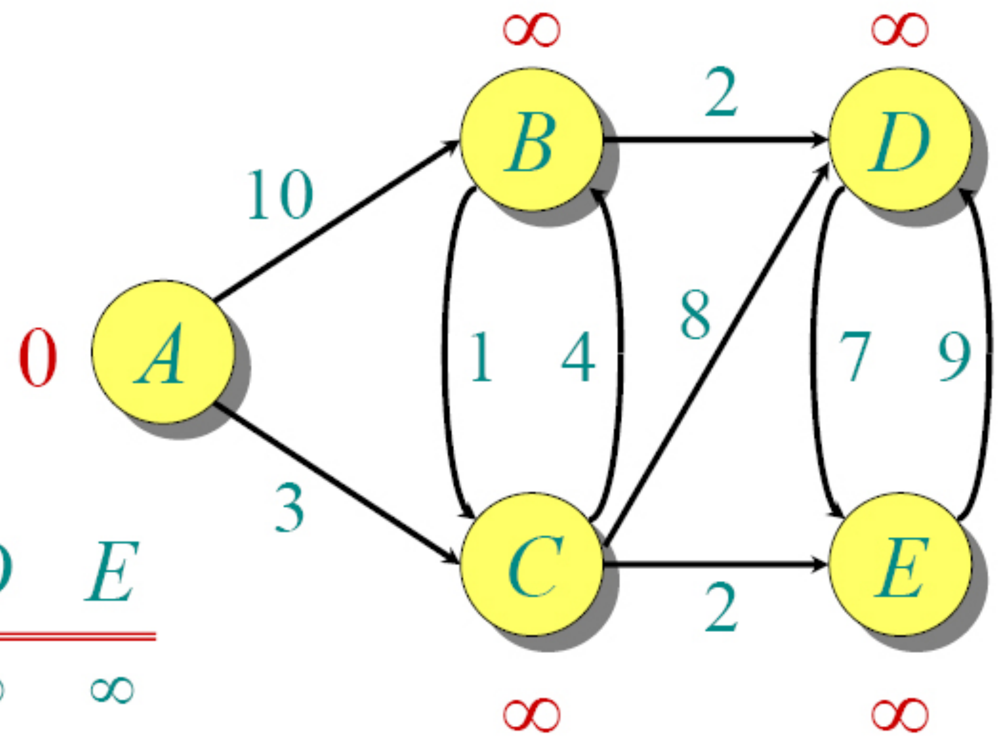
  … but guarantees the optimum!

Dijkstra's algorithm

```
dist[s] ←0                      (distance to source vertex is zero)
for  all v ∈ V–{s}
     do  dist[v] ←∞             (set all other distances to infinity)
S←Ø                             (S, the set of visited vertices is initially empty)
Q←V                             (Q, the queue initially contains all vertices)
while Q ≠Ø                      (while the queue is not empty)
do   u ← mindistance(Q,dist)    (select e ∈ Q with the min. distance)
    S←S∪{u}                     (add u to list of visited vertices)
     for all v ∈ neighbors[u]
          do  if  dist[v] > dist[u] + w(u, v)       (if new shortest path found)
                   then  d[v] ←d[u] + w(u, v)        (set new value of shortest path)
                                (if desired, add traceback code)
```

Dijkstra Animated Example

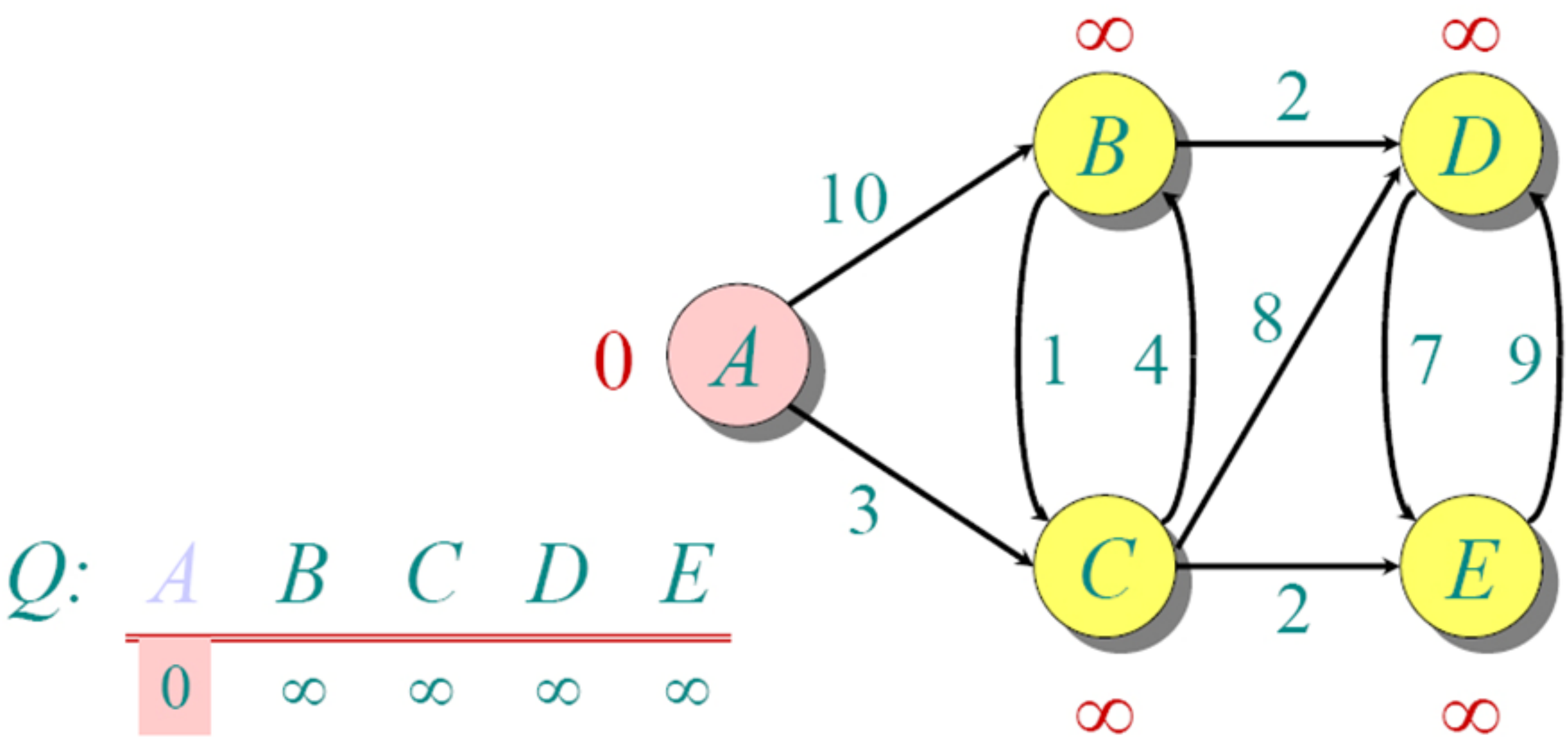
Initialize:



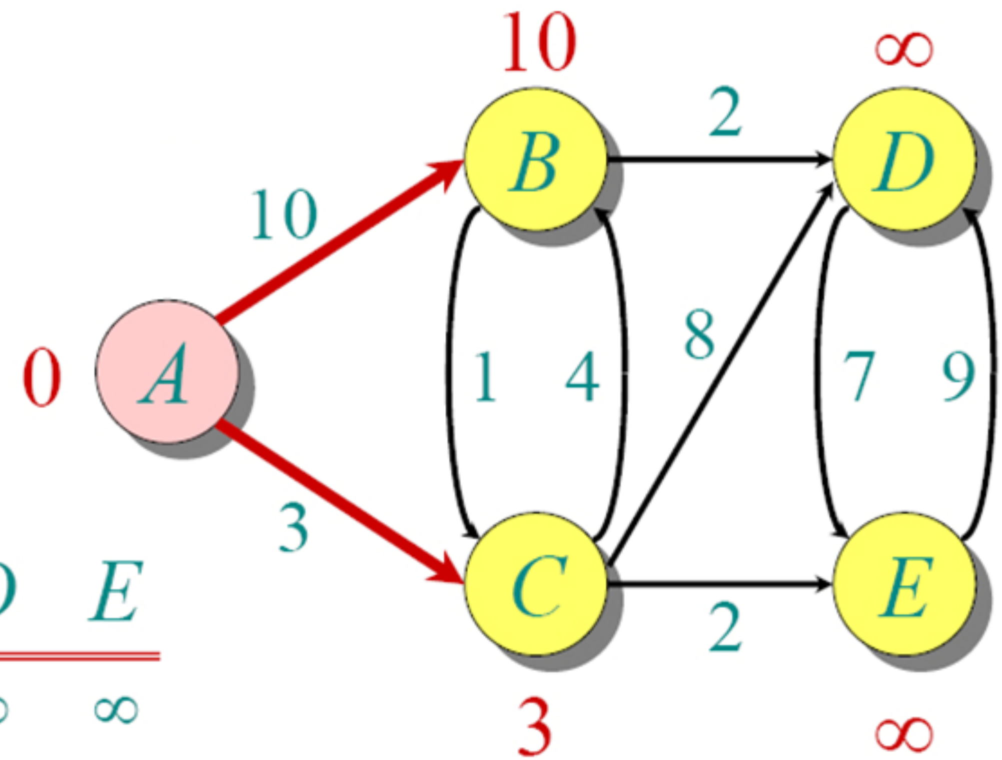
Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: {}

Dijkstra Animated Example



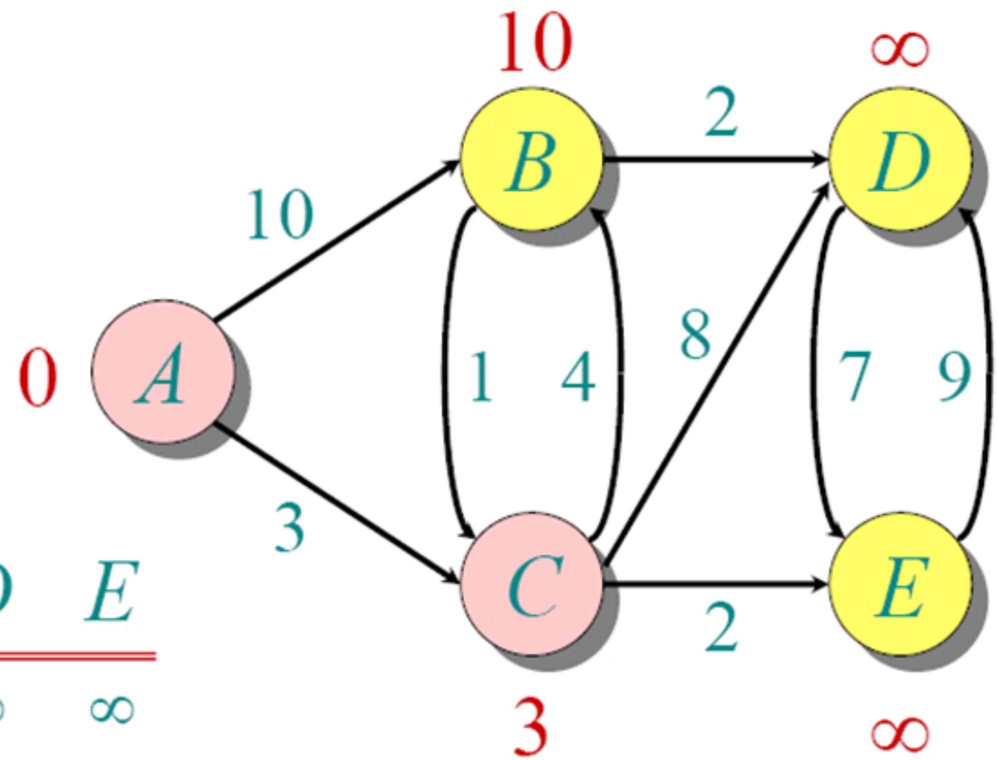
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A }

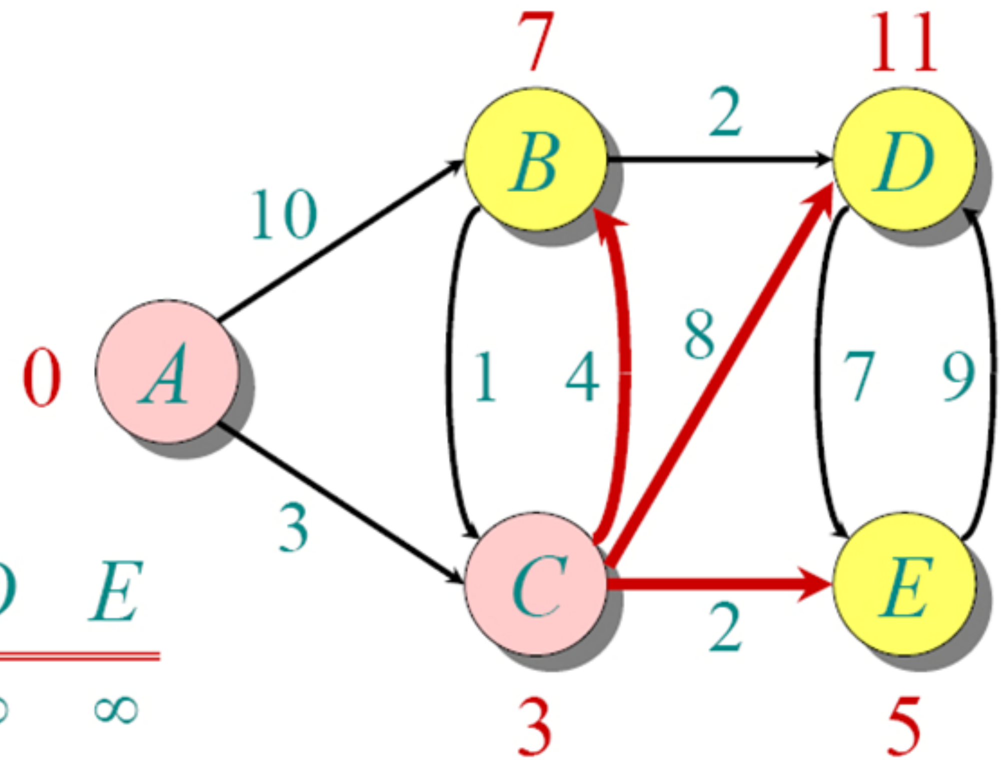
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A, C }

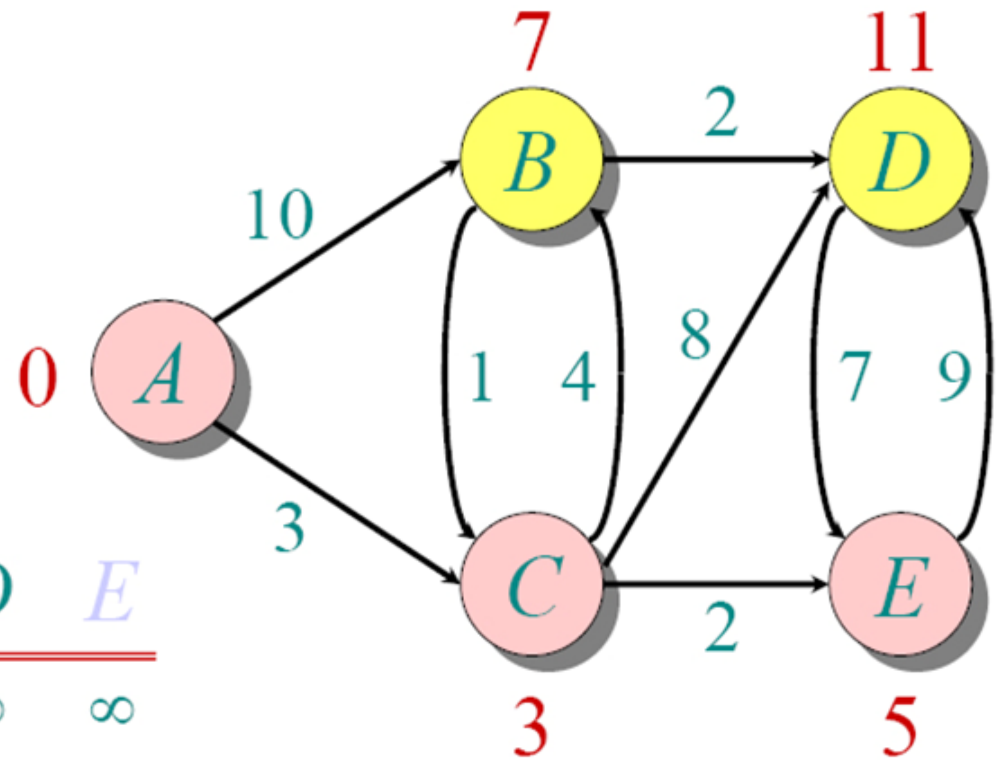
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { A, C }

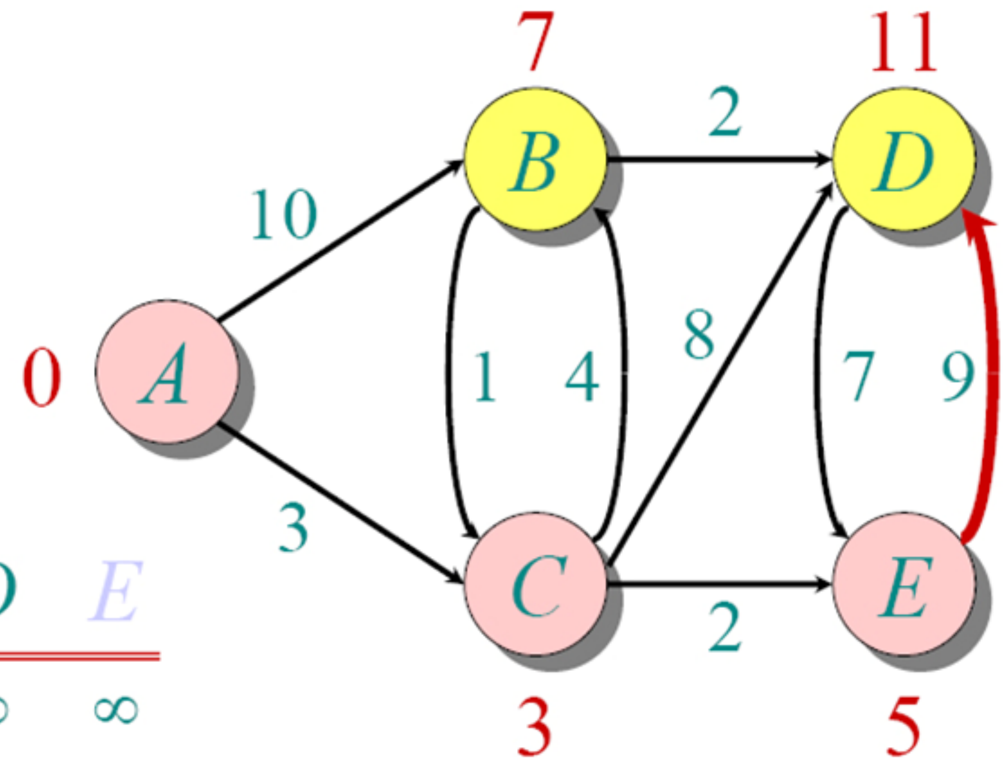
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { A, C, E }

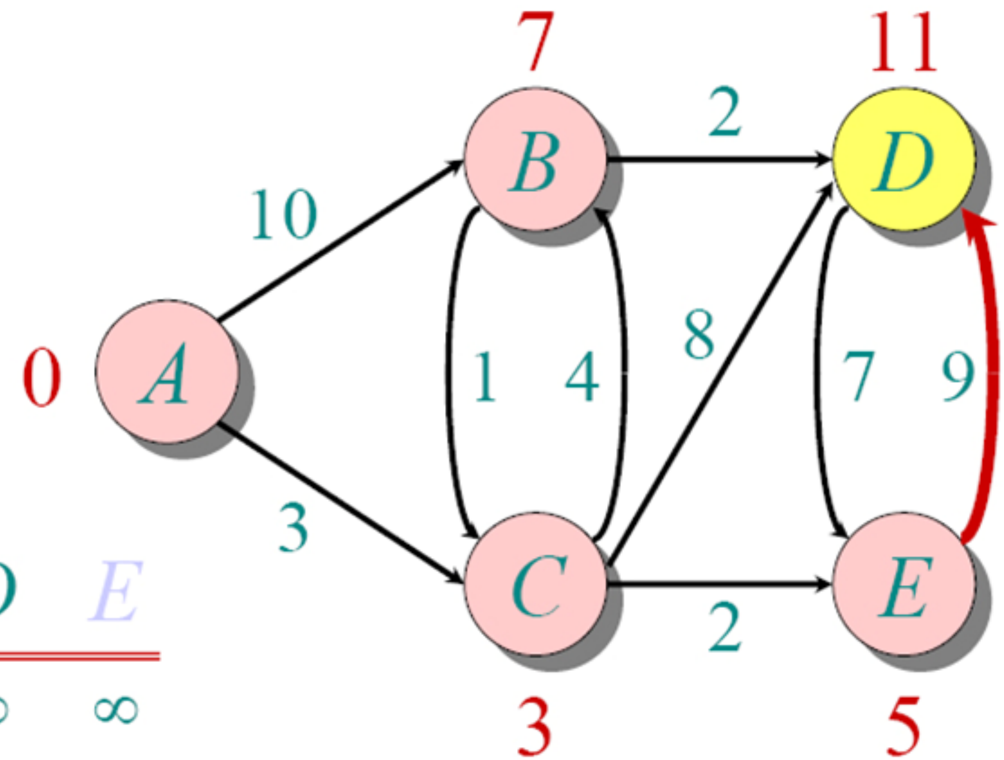
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E }

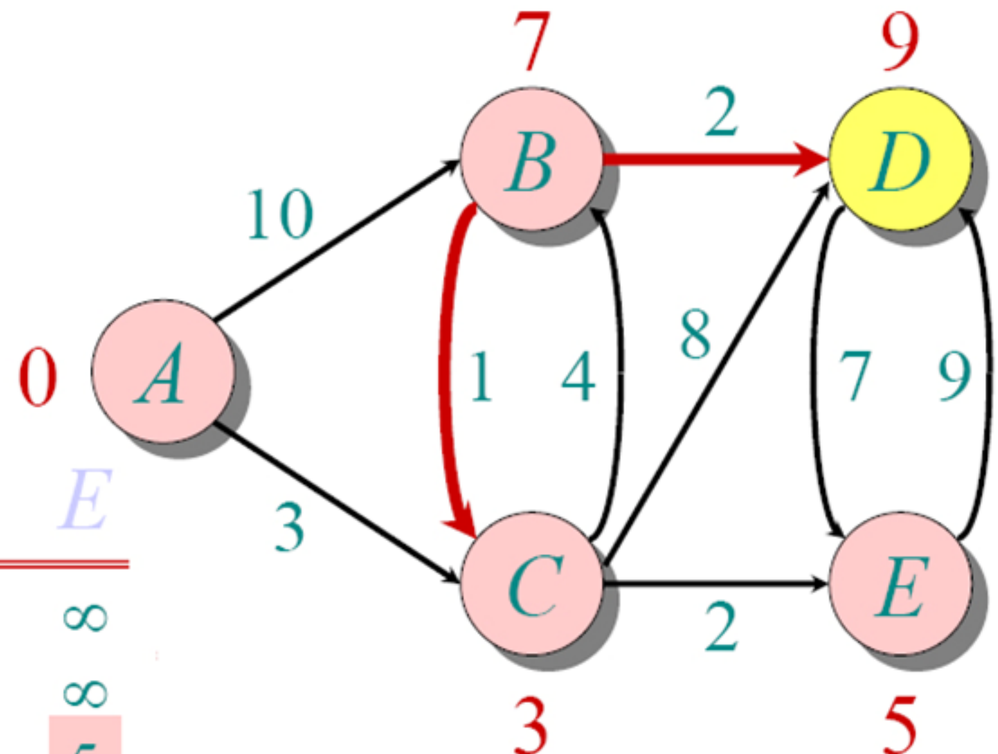
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E, B }

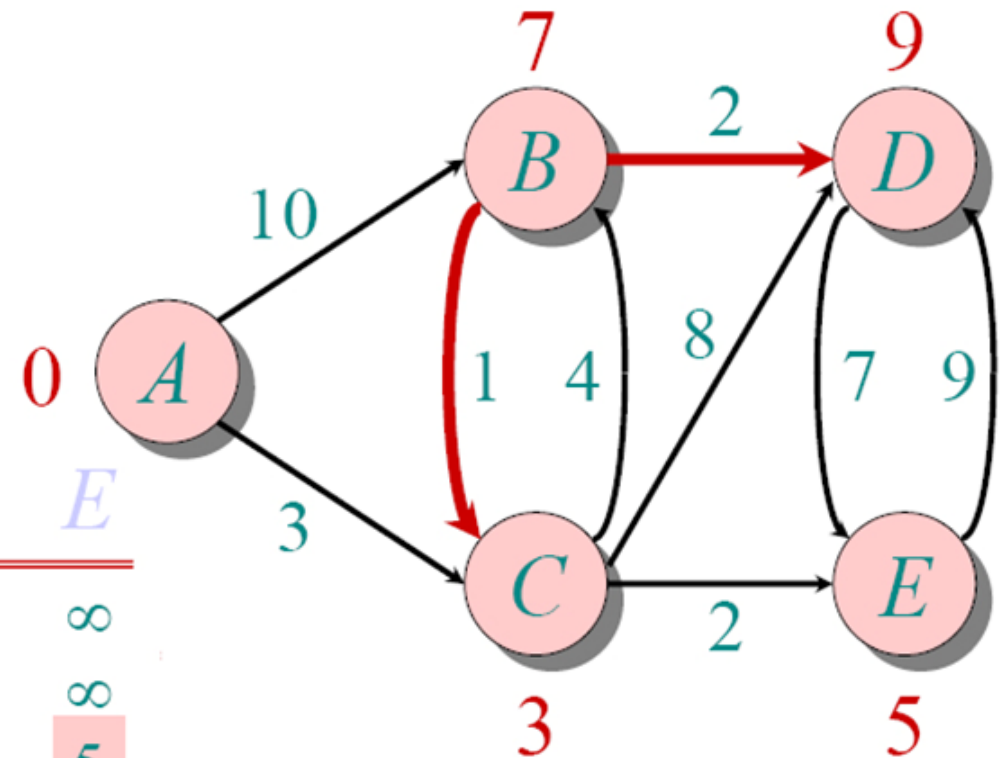
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { *A*, *C*, *E*, *B* }

Dijkstra Animated Example



Q:

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	

S: { A, C, E, B, D }

Why it works

- A formal proof would take longer than this presentation, but we can understand how the argument works intuitively
  - Think of Djikstra's algorithm as a water-filling algorithm
  - Remember that all edge's weights are positive

Dijkstra efficiency

- The simplest implementation is:

$$O(E + V^2)$$

- But it can be implemented more efficently:

$$O(E + V \cdot \log V)$$

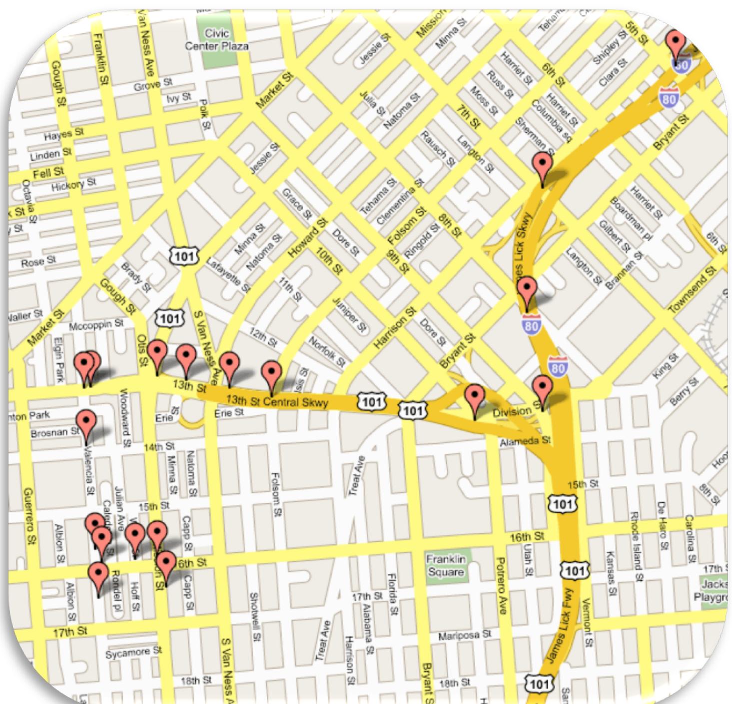
Floyd–Warshall: $O(V^3)$
Bellman-Ford-Moore : $O(V \cdot E)$

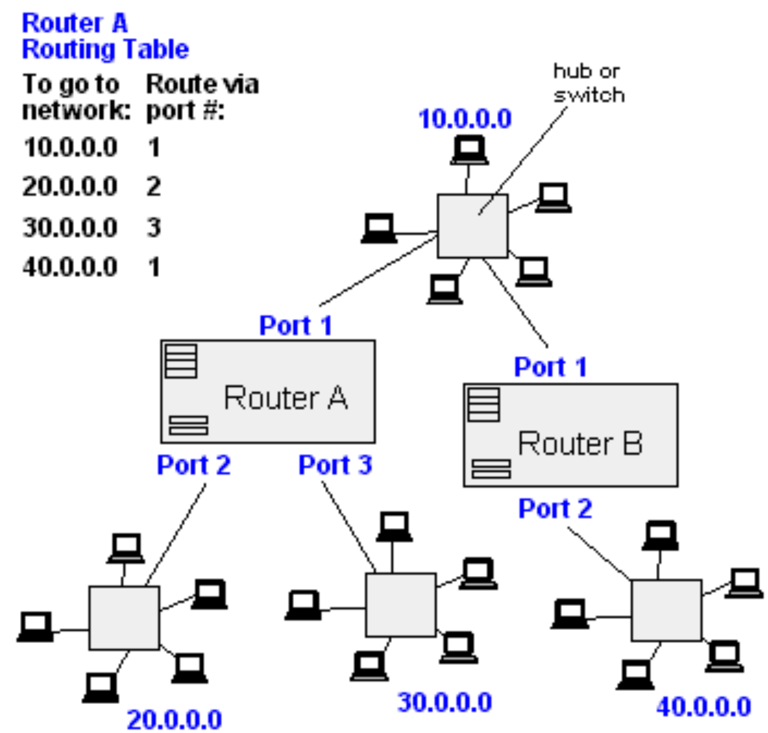
Applications

- Dijkstra's algorithm calculates the shortest path to every vertex from vertex **s** (SS-SP)
- It is about as computationally expensive to calculate the shortest path from vertex u to every vertex using Dijkstra's as it is to calculate the shortest path to some particular vertex **t**
- Therefore, anytime we want to know the optimal path to some other vertex **t** from a determined origin **s**, we can use Dijkstra's algorithm (and stop as soon **t** exit from **Q**)

Applications

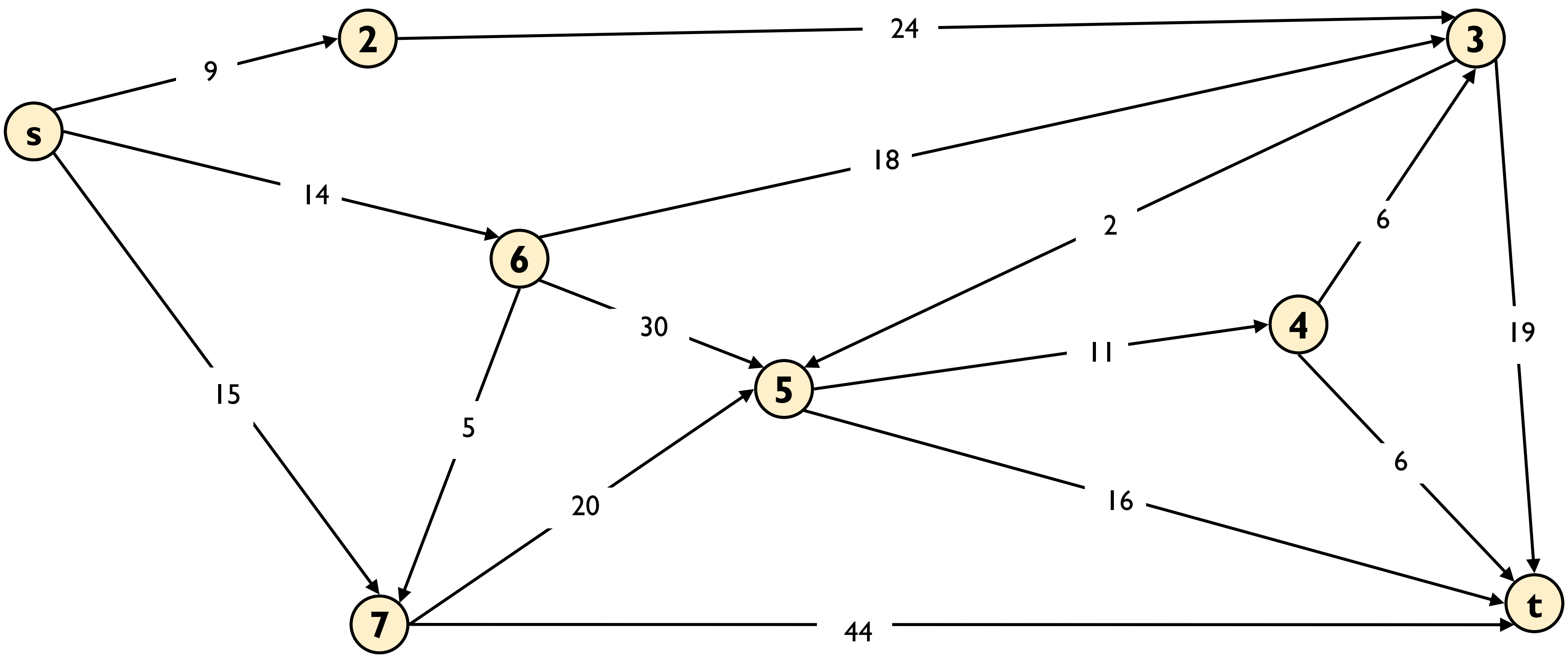
- Traffic Information Systems are most prominent use
- Mapping (Map Quest, Google Maps)
- Routing Systems





Dijkstra's Shortest Path Algorithm

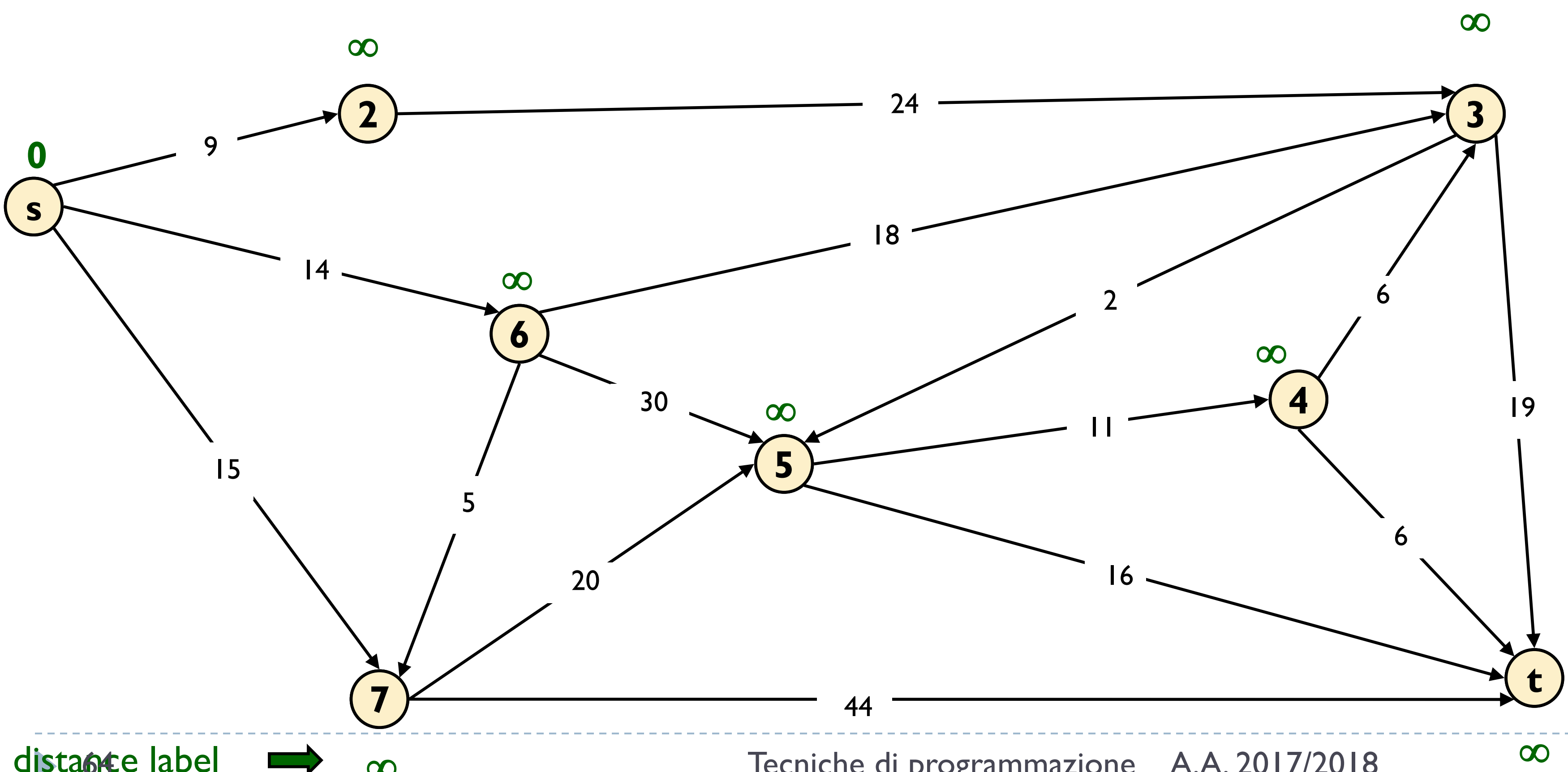
- Find shortest path from **s** to **t**



Dijkstra's Shortest Path Algorithm

S = { }

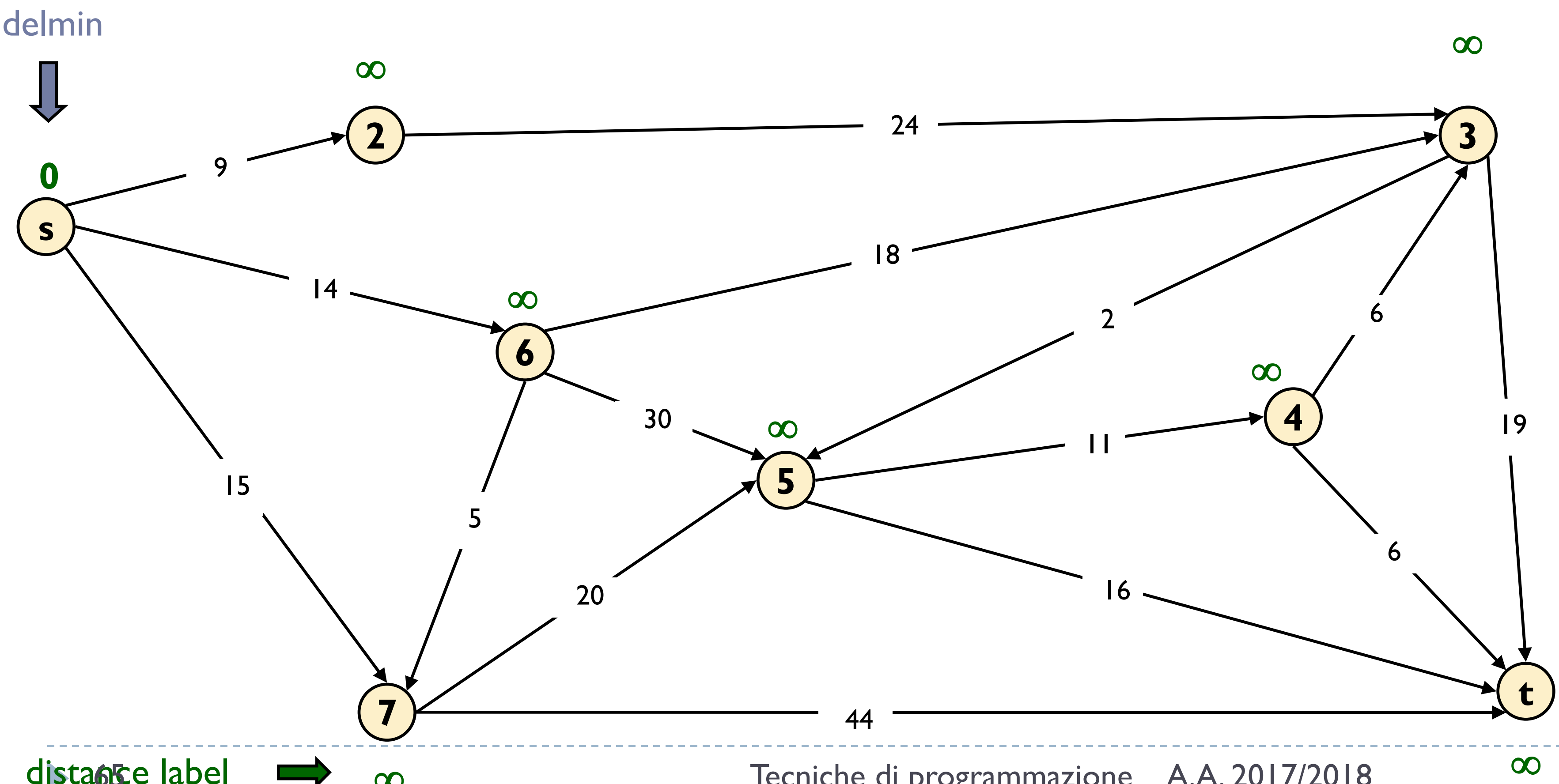
Q = { s, 2, 3, 4, 5, 6, 7, t }



Dijkstra's Shortest Path Algorithm

S = { }

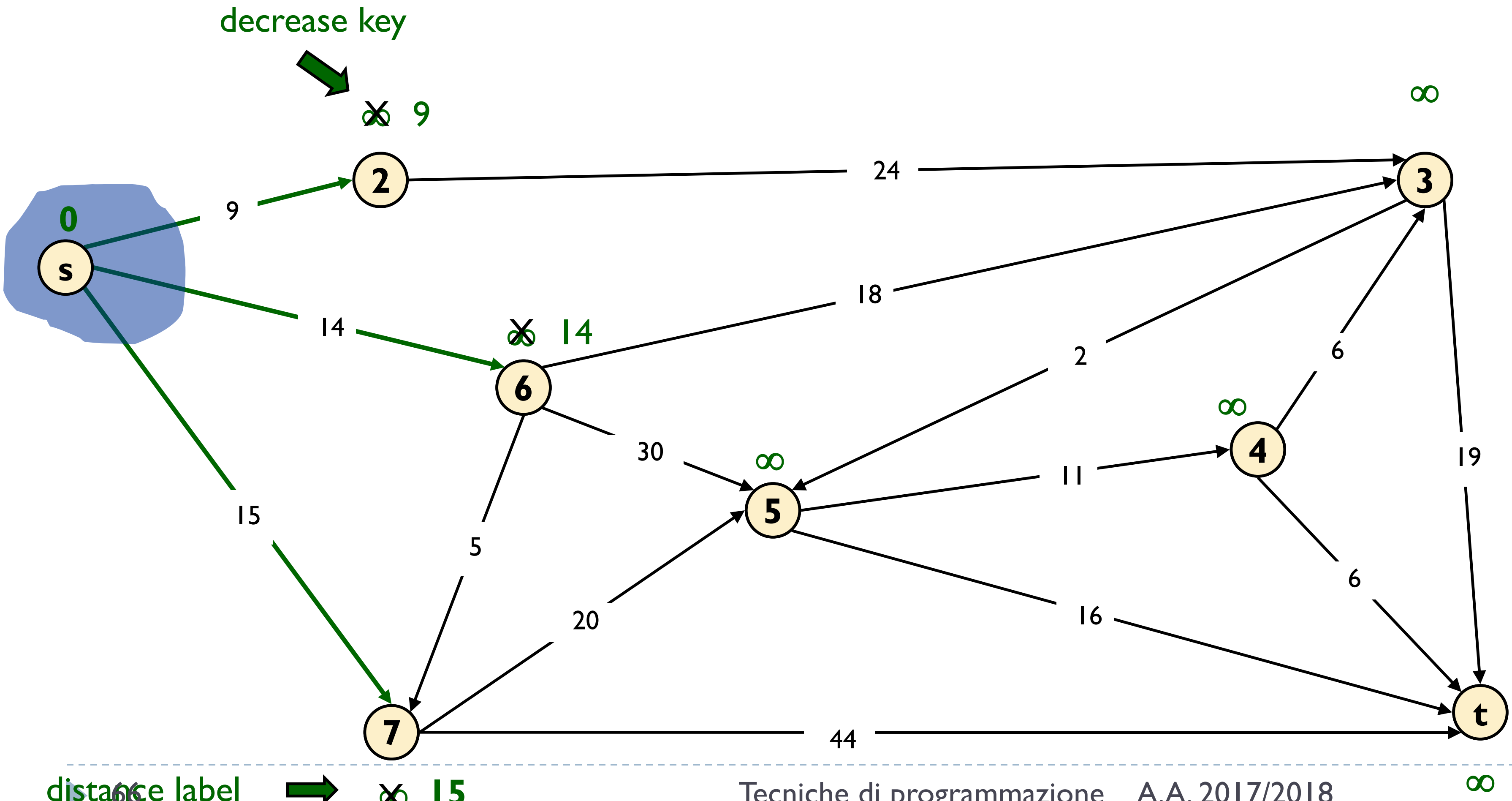
Q = { s, 2, 3, 4, 5, 6, 7, t }



Dijkstra's Shortest Path Algorithm

S = { s }

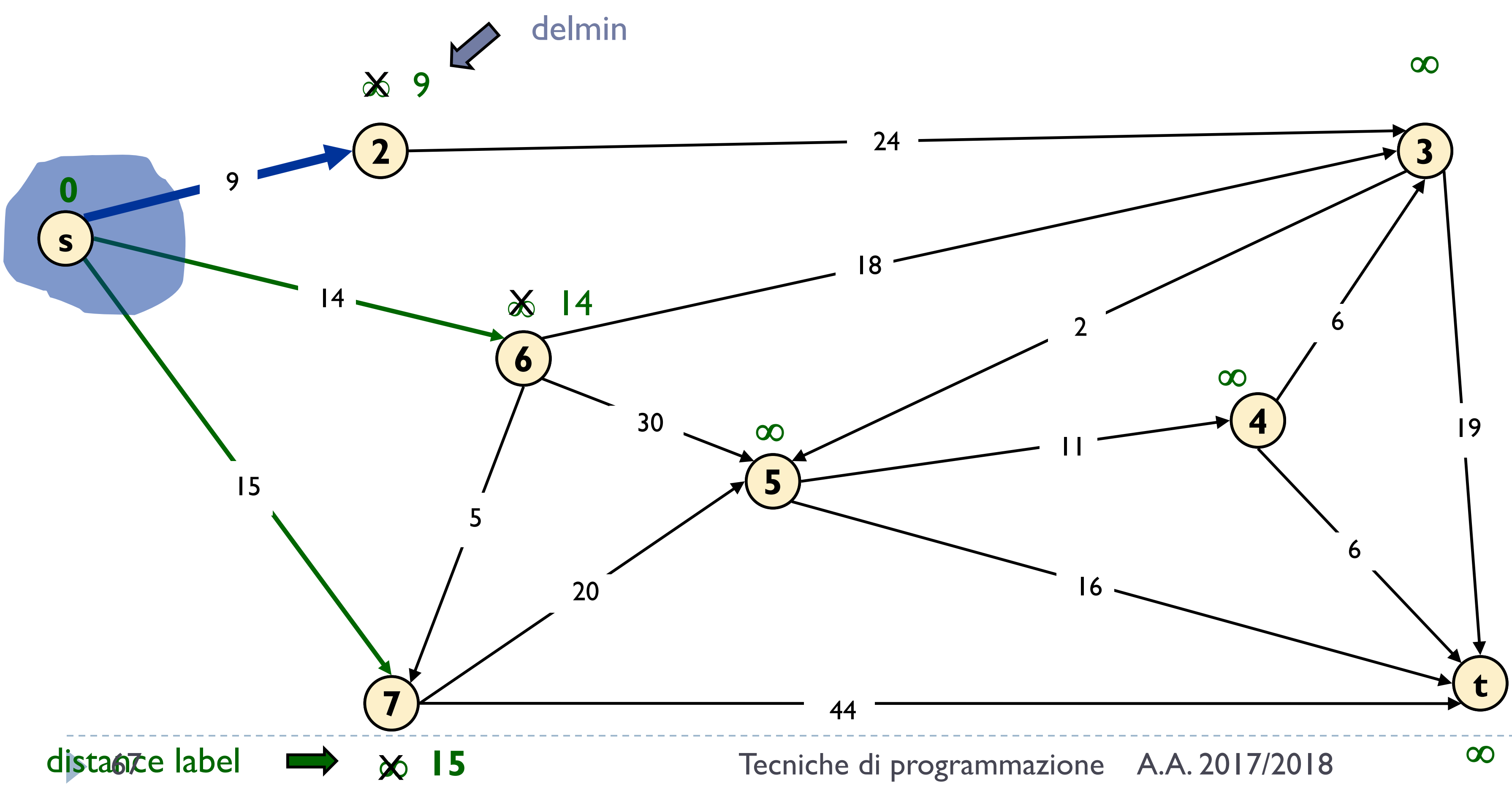
Q = { 2, 3, 4, 5, 6, 7, t }



Dijkstra's Shortest Path Algorithm

S = { s }

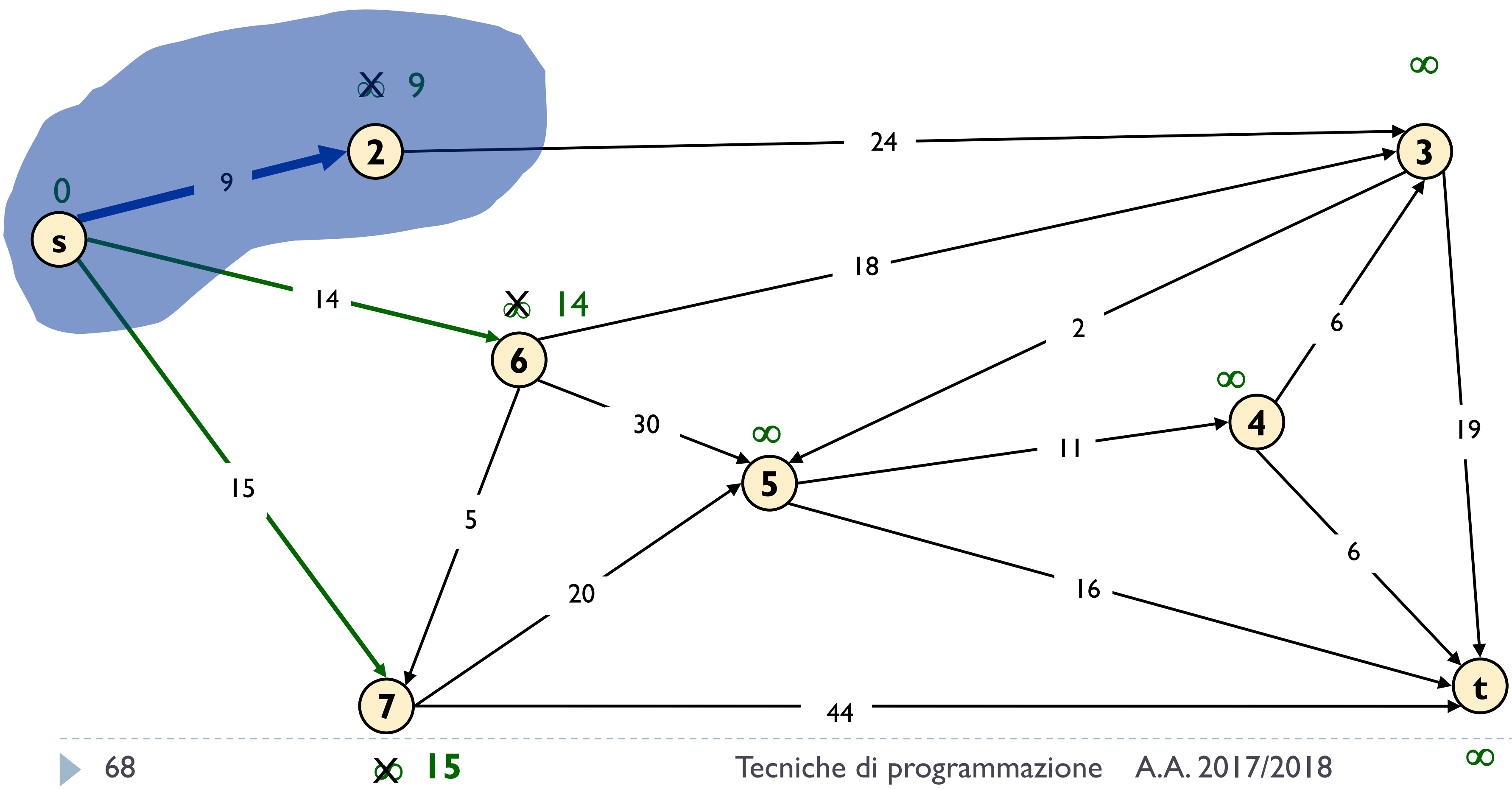
Q = { 2, 3, 4, 5, 6, 7, t }



Dijkstra's Shortest Path Algorithm

S = { s, 2 }

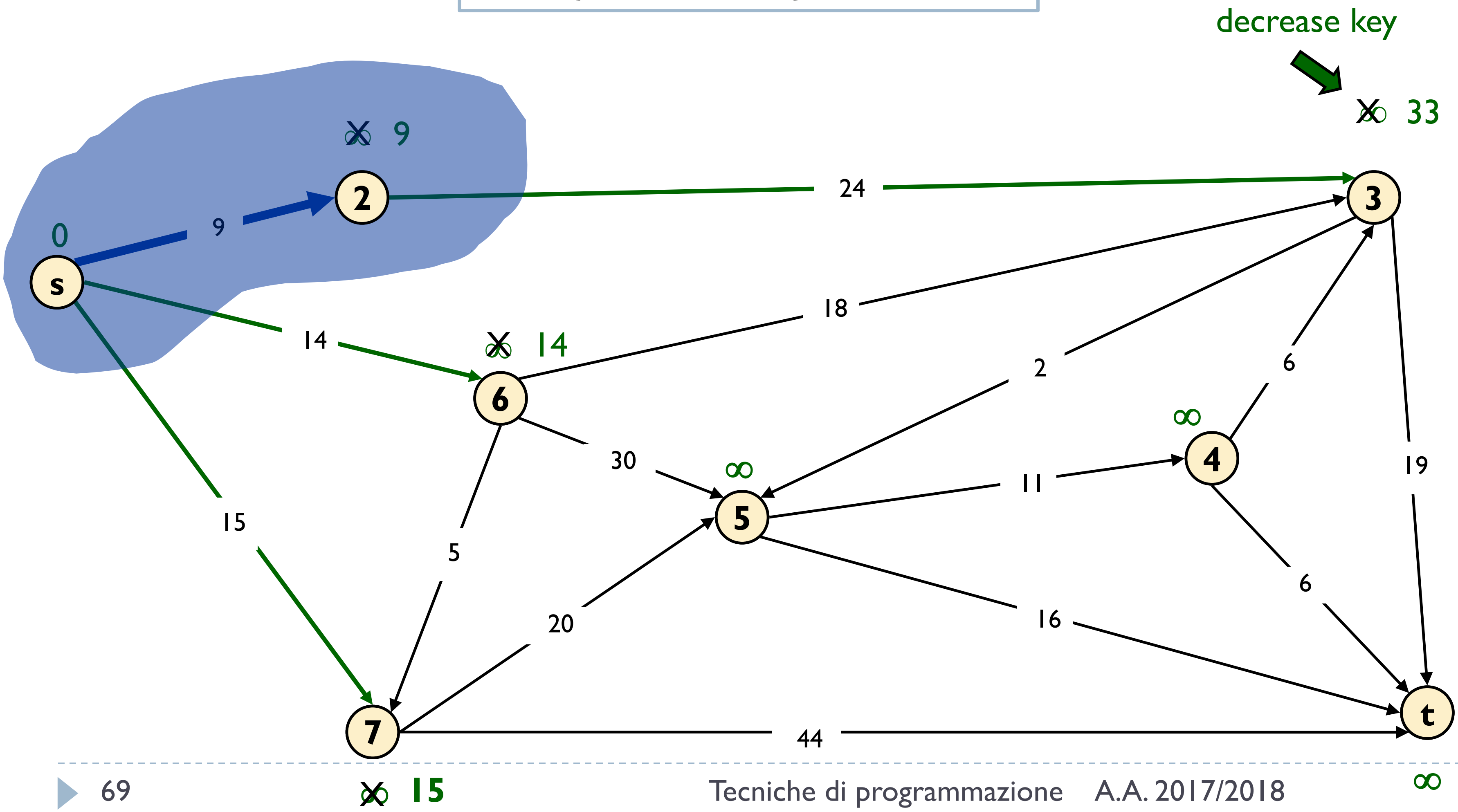
Q = { 3, 4, 5, 6, 7, t }



Dijkstra's Shortest Path Algorithm

S = { s, 2 }

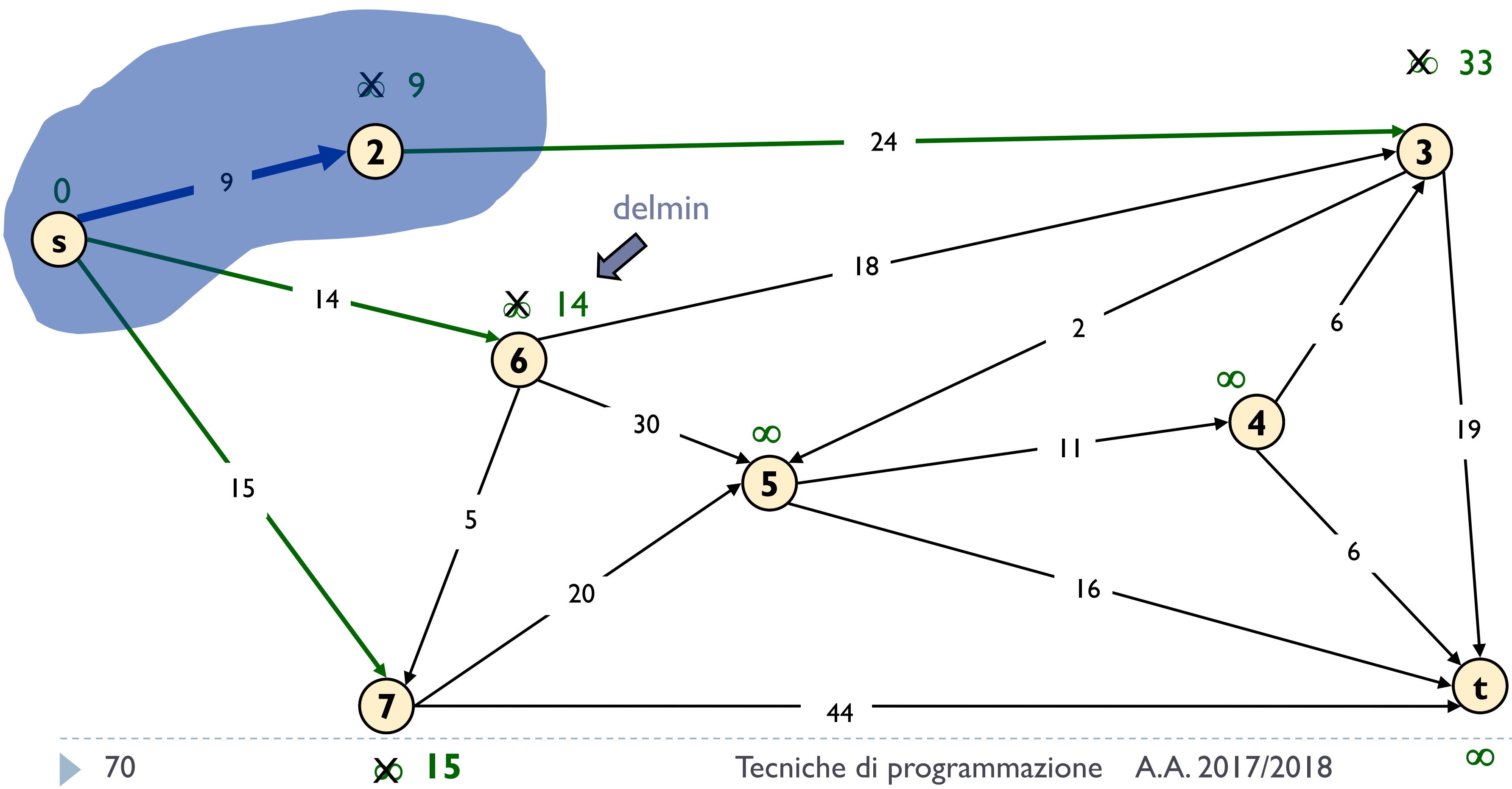
Q = { 3, 4, 5, 6, 7, t }



Dijkstra's Shortest Path Algorithm

S = { s, 2 }

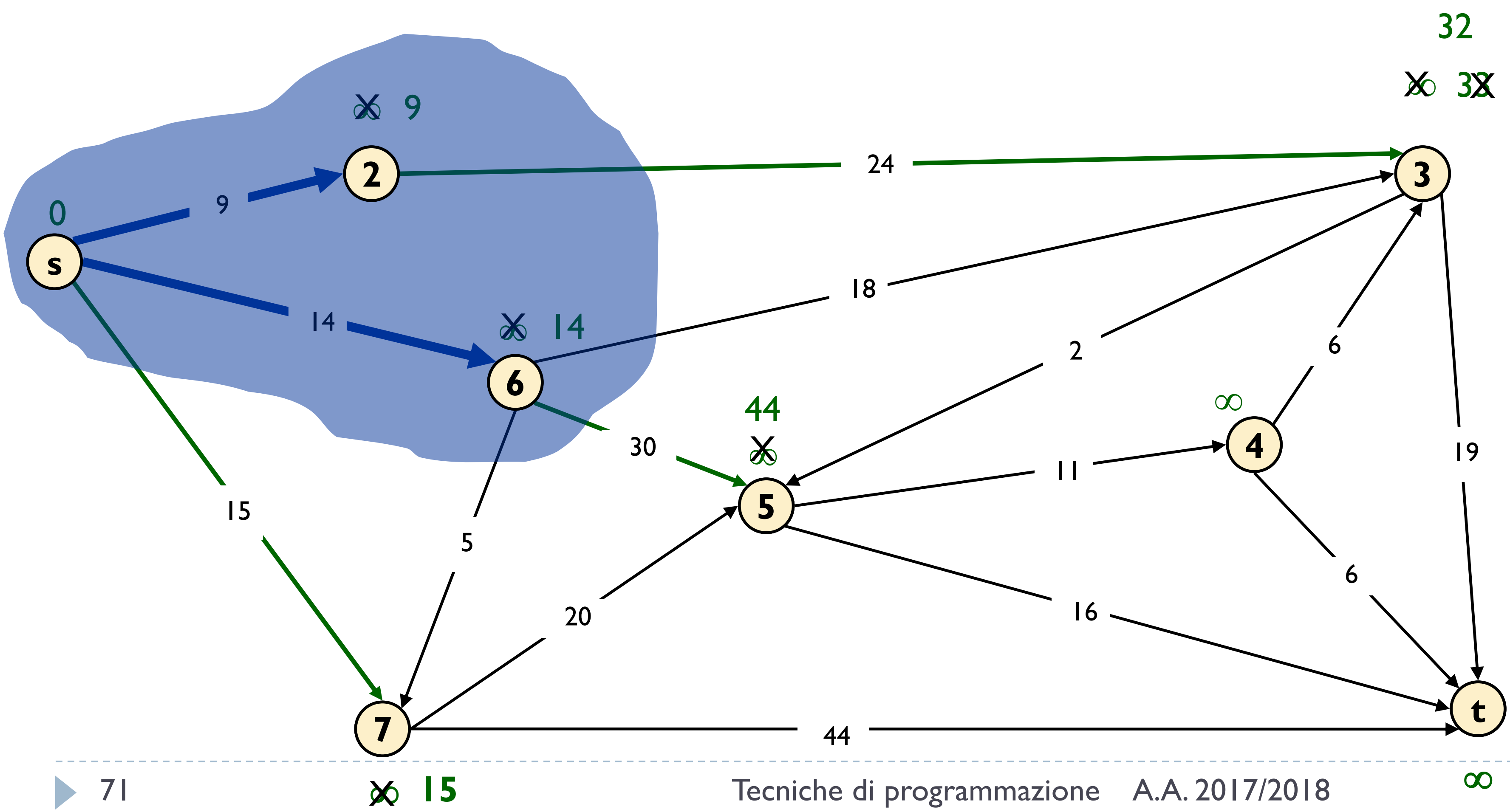
Q = { 3, 4, 5, 6, 7, t }



Dijkstra's Shortest Path Algorithm

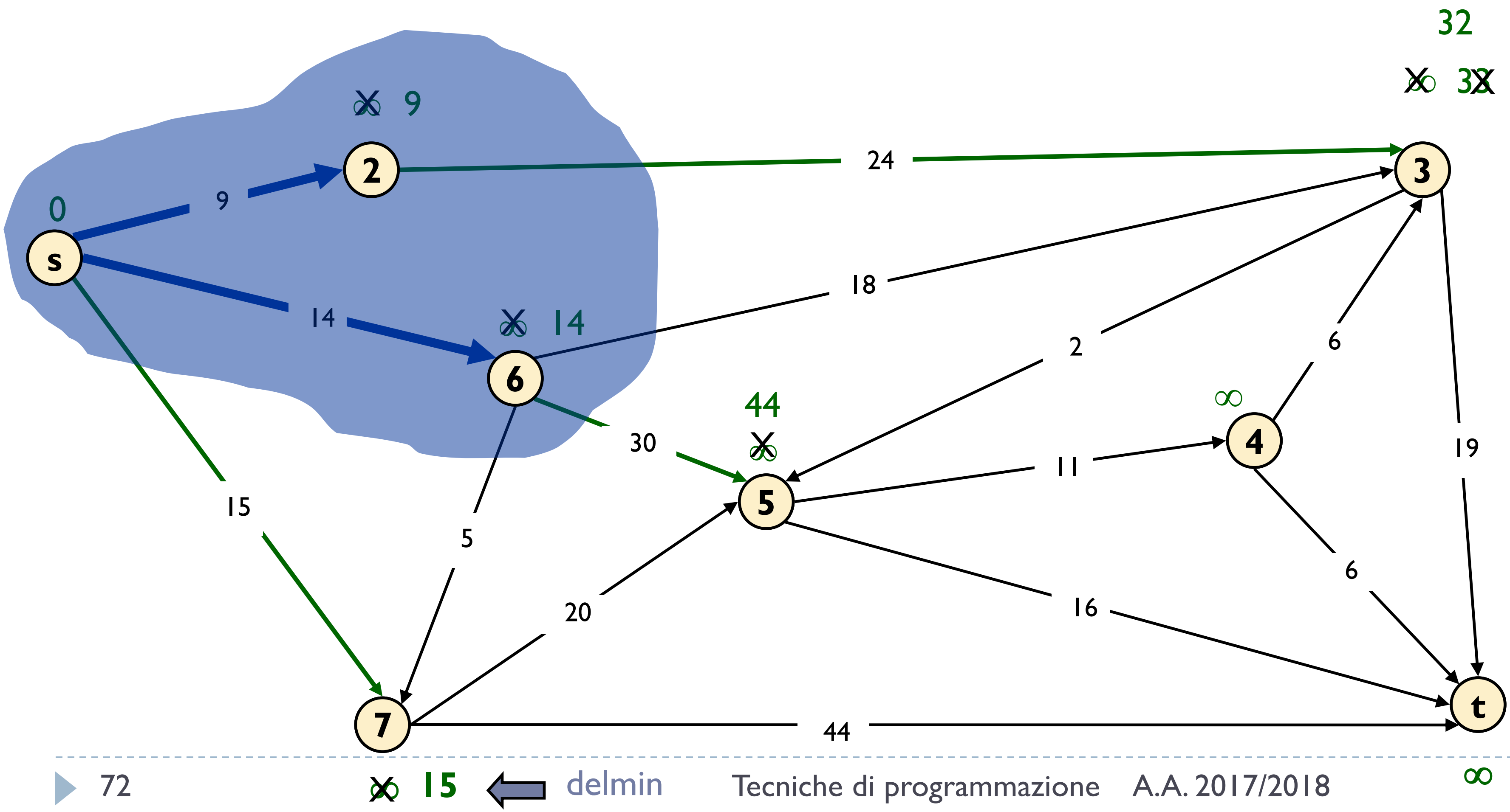
S = { s, 2, 6 }

Q = { 3, 4, 5, 7, t }



Dijkstra's Shortest Path Algorithm

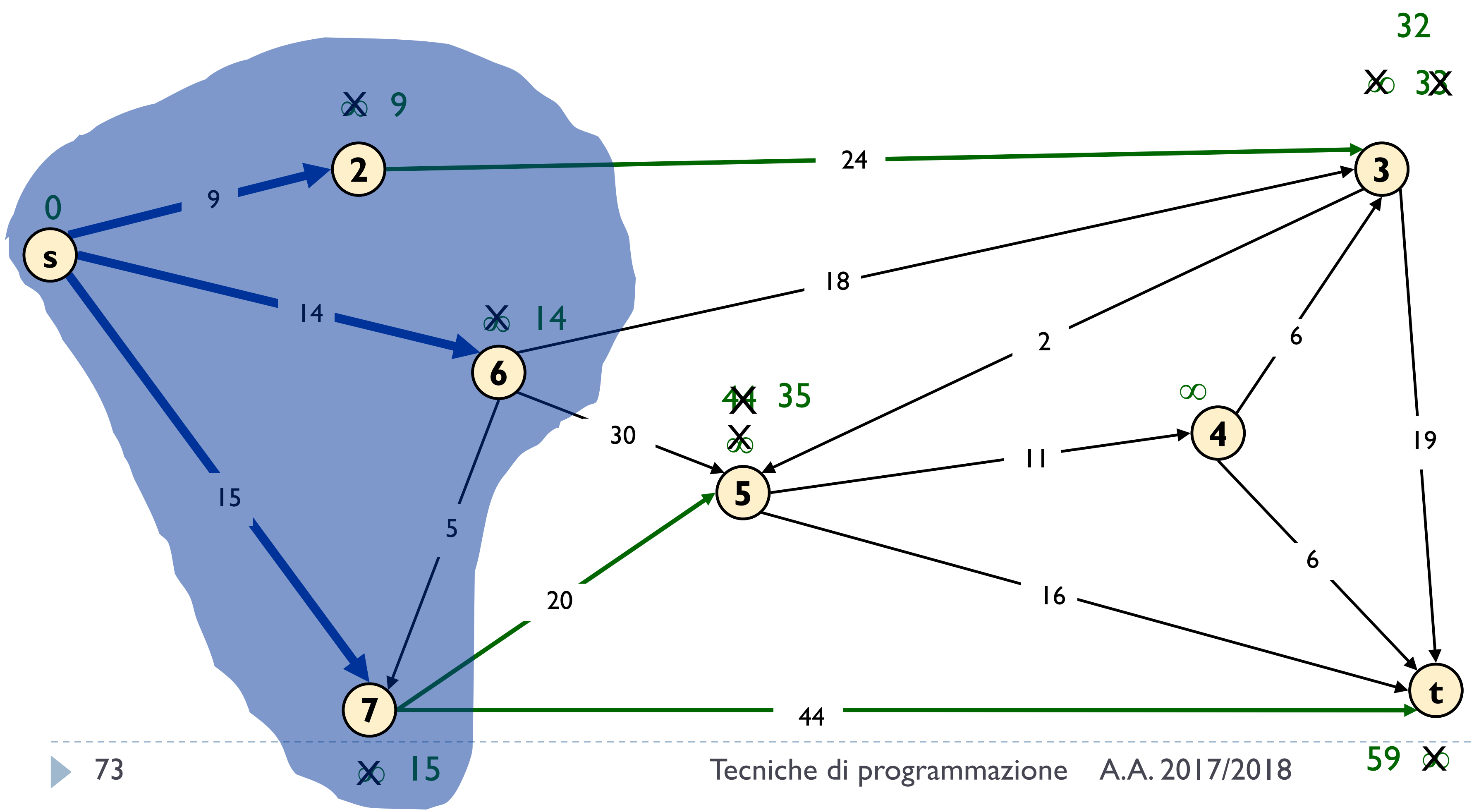
S = { s, 2, 6 }
Q = { 3, 4, 5, 7, t }



Dijkstra's Shortest Path Algorithm

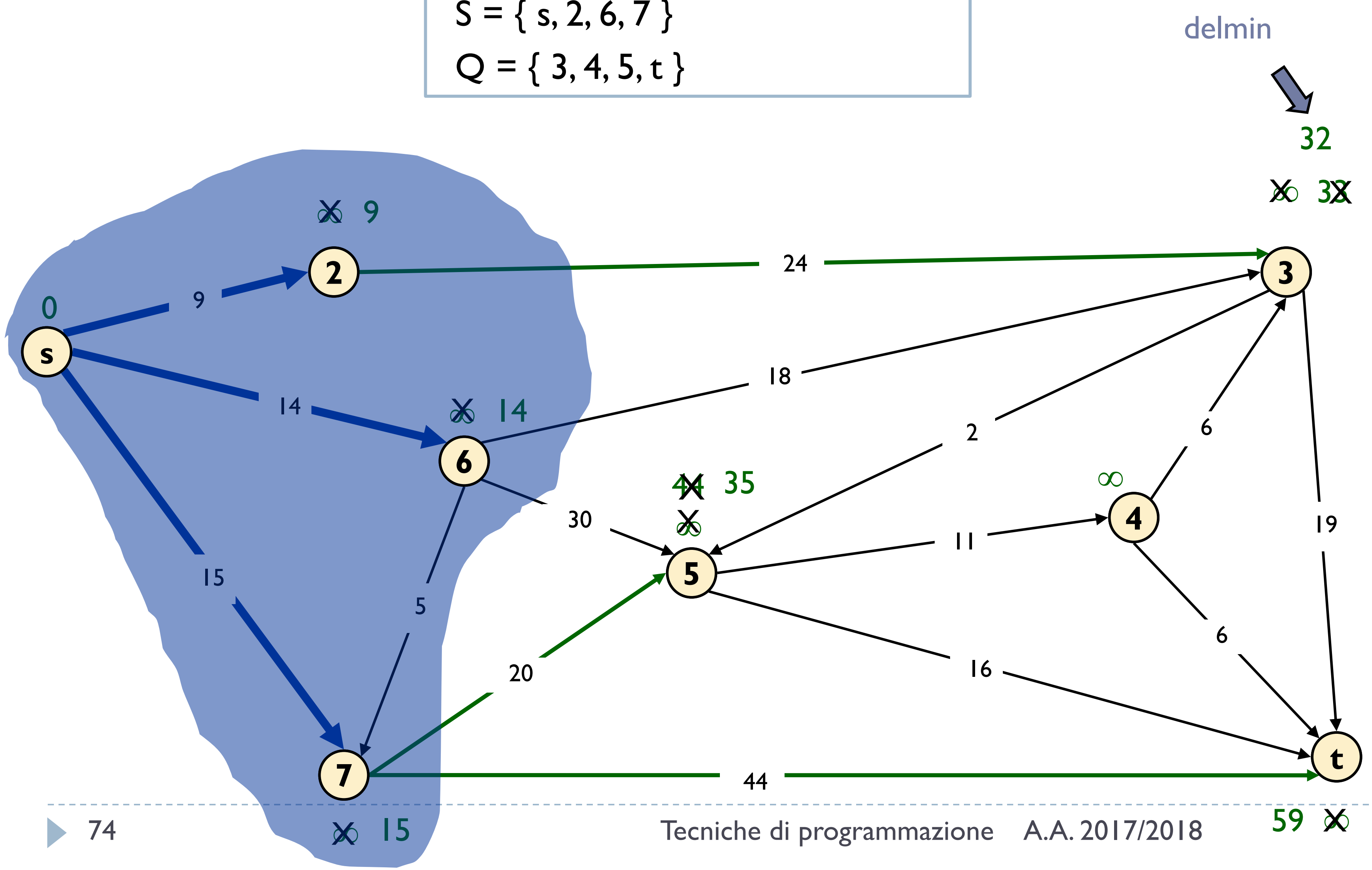
S = { s, 2, 6, 7 }

Q = { 3, 4, 5, t }



Dijkstra's Shortest Path Algorithm

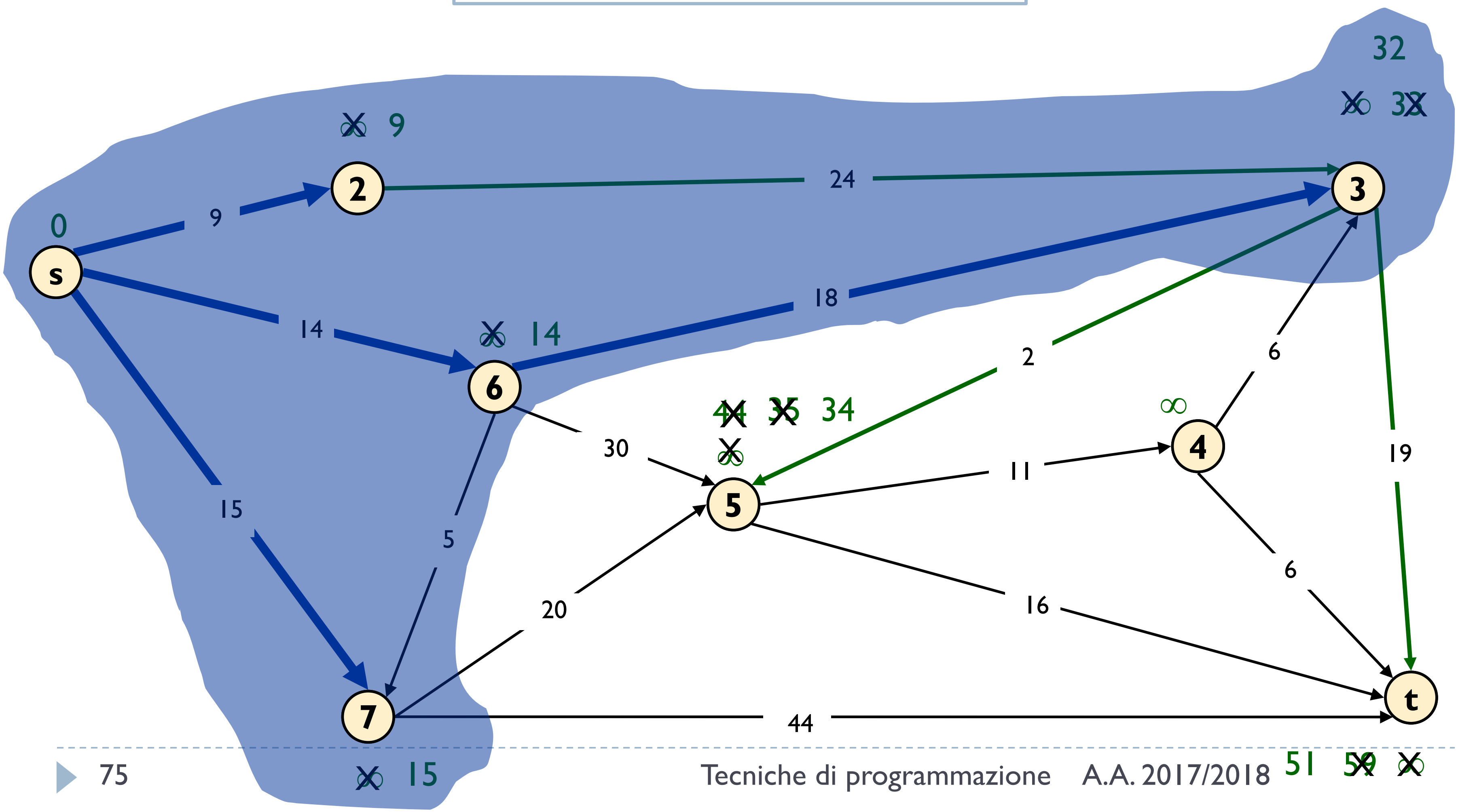
S = { s, 2, 6, 7 }
Q = { 3, 4, 5, t }



Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 6, 7 }

Q = { 4, 5, t }



Dijkstra's Shortest Path Algorithm

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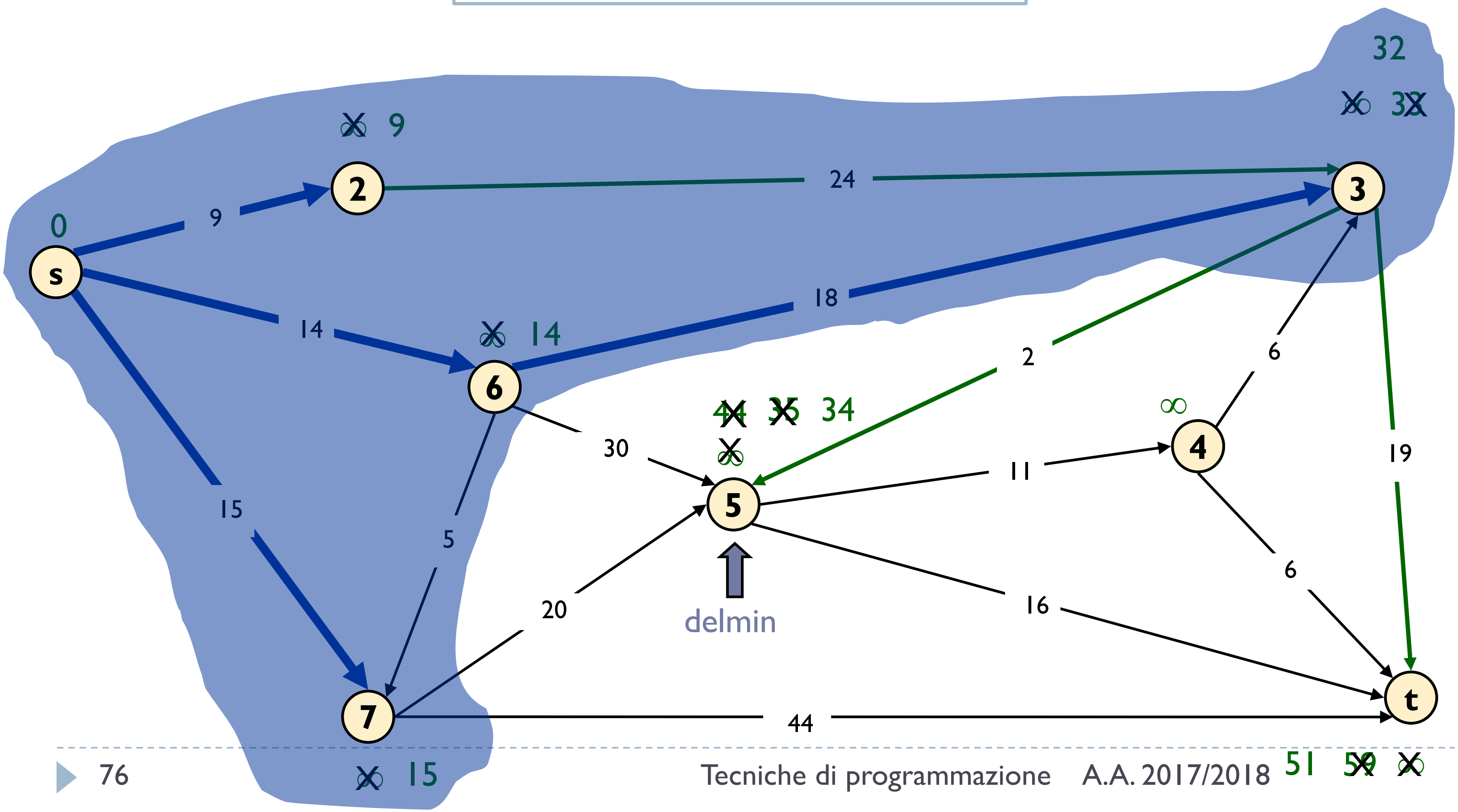
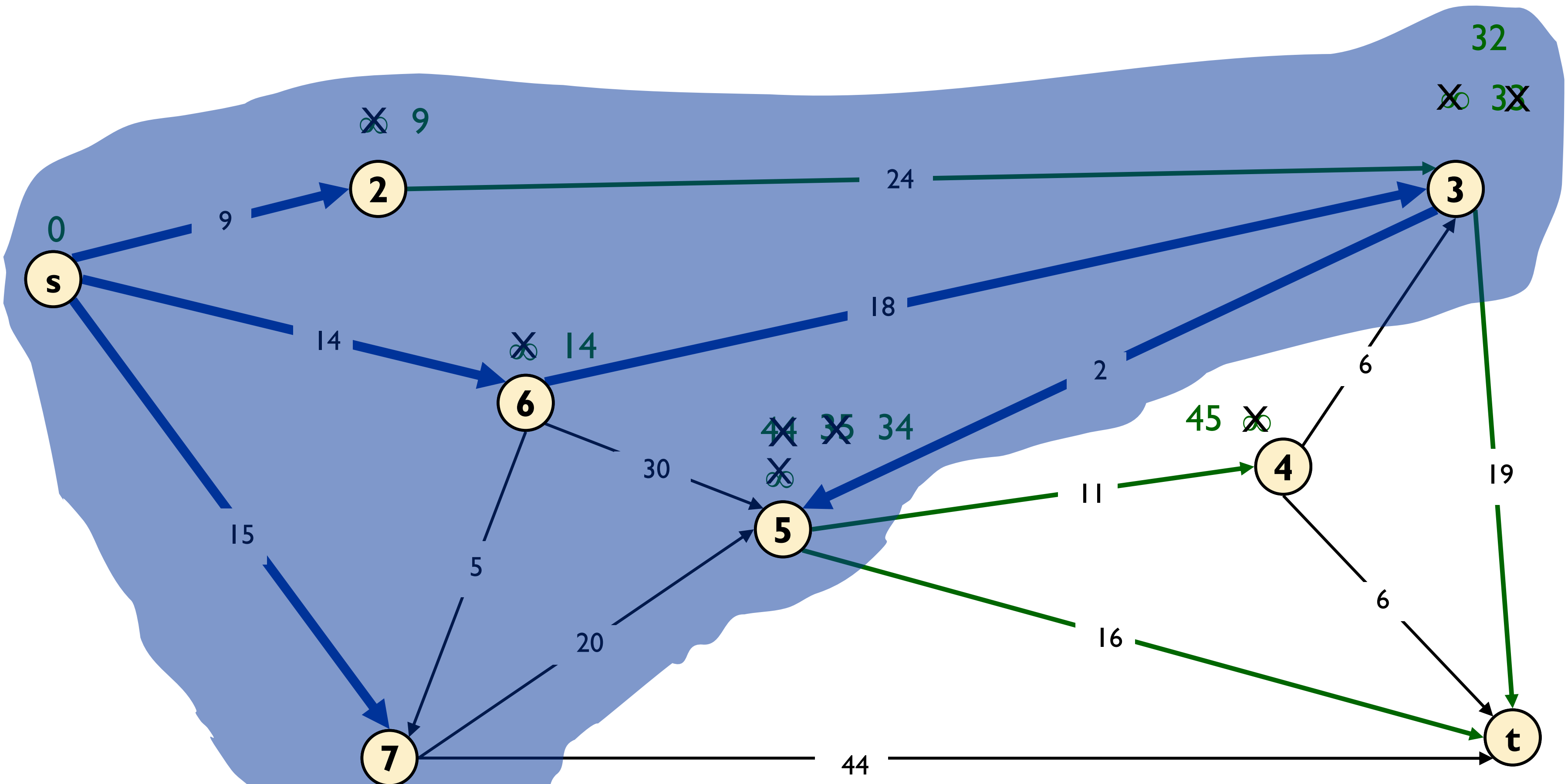
Q = { 4, 5, t }



Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 5, 6, 7 }

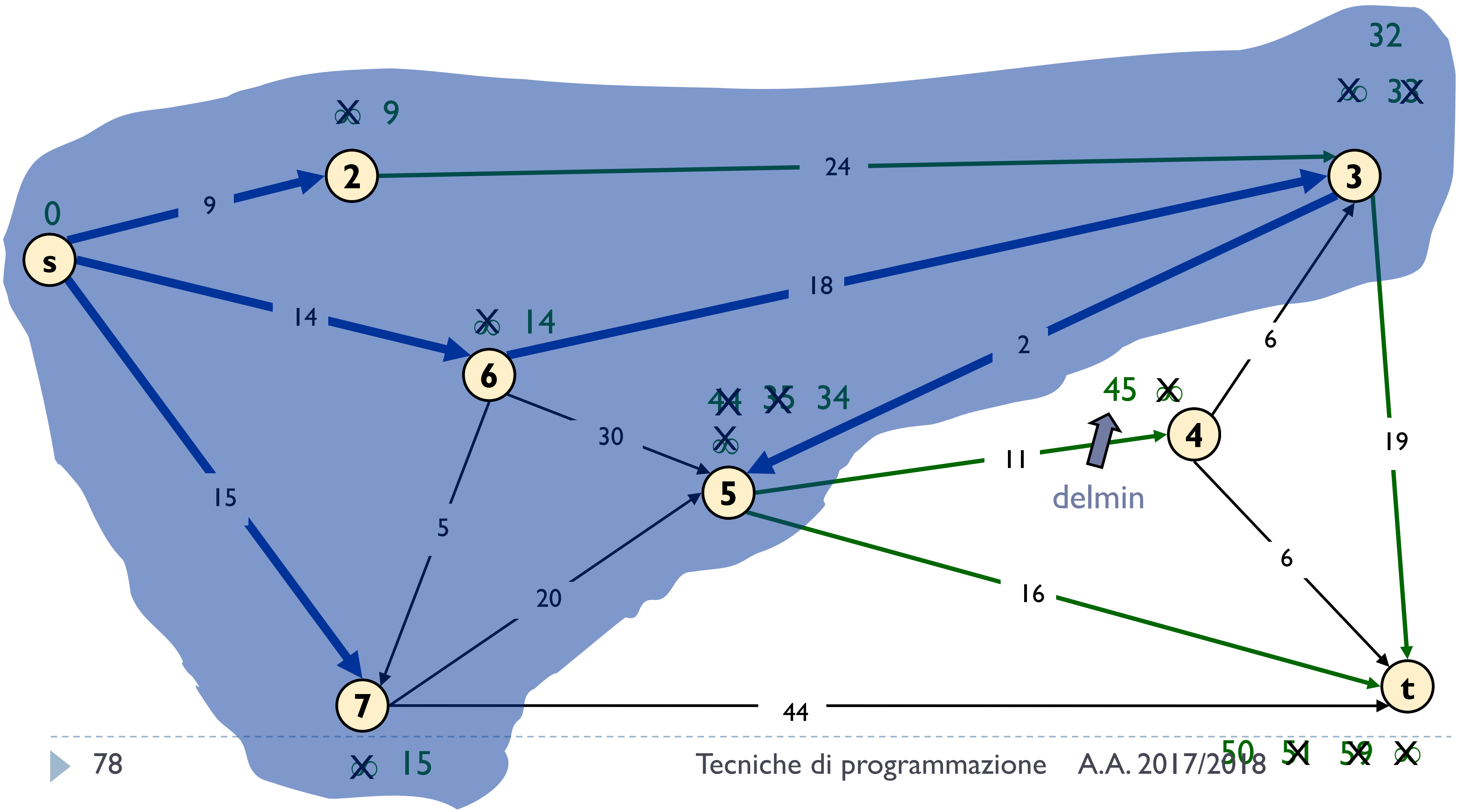
Q = { 4, t }



Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 5, 6, 7 }

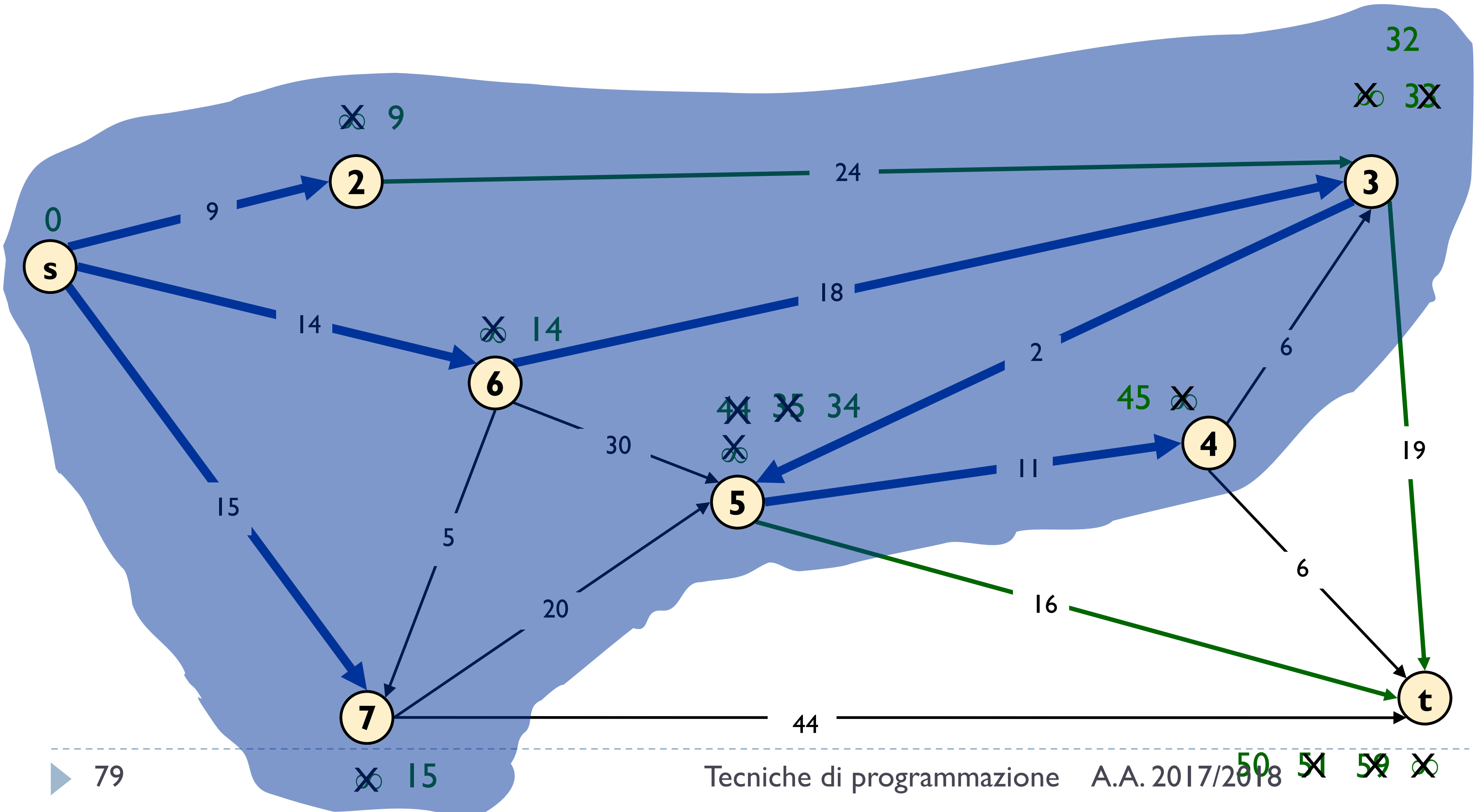
Q = { 4, t }



Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 4, 5, 6, 7 }

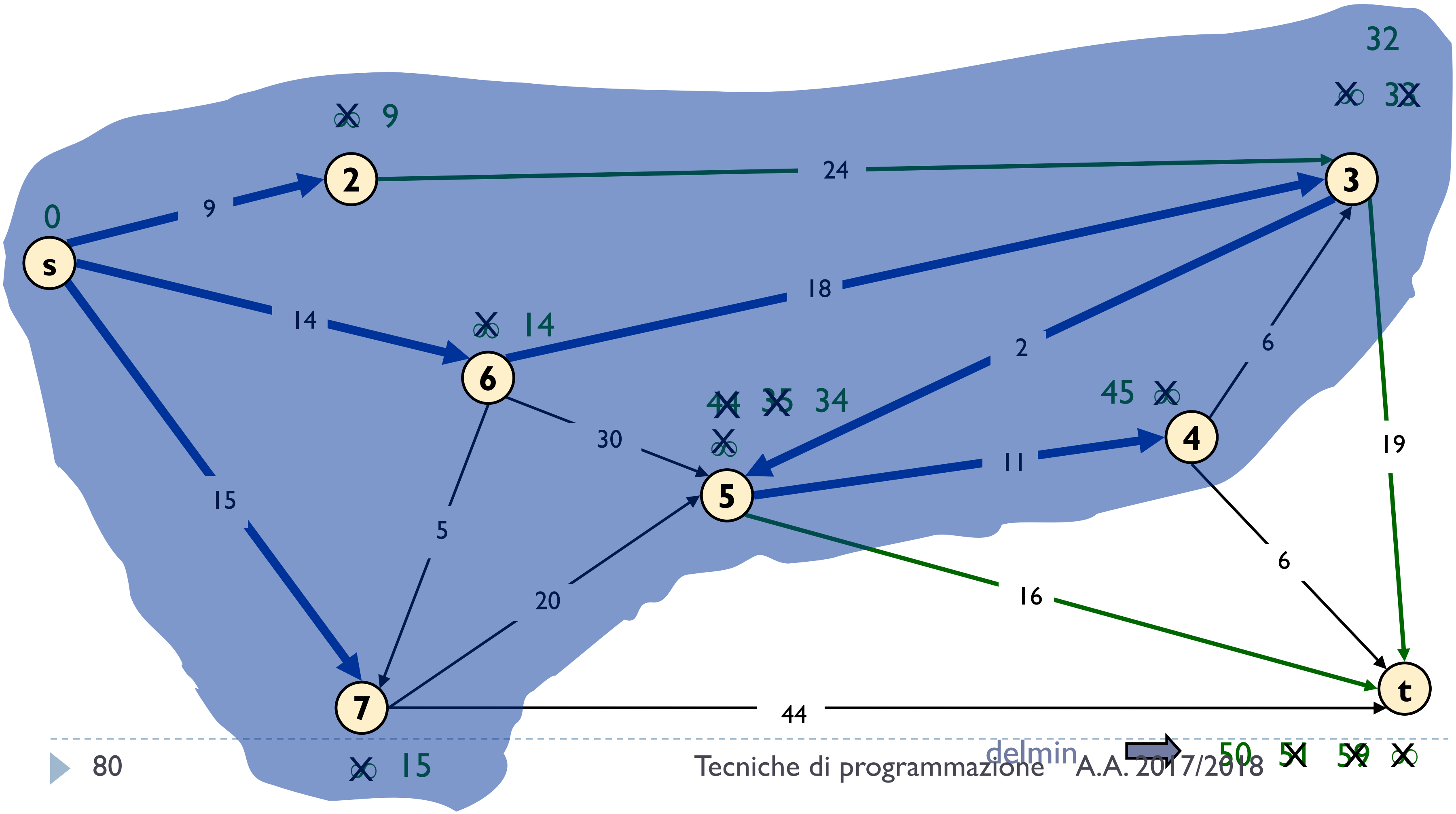
Q = { t }



Dijkstra's Shortest Path Algorithm

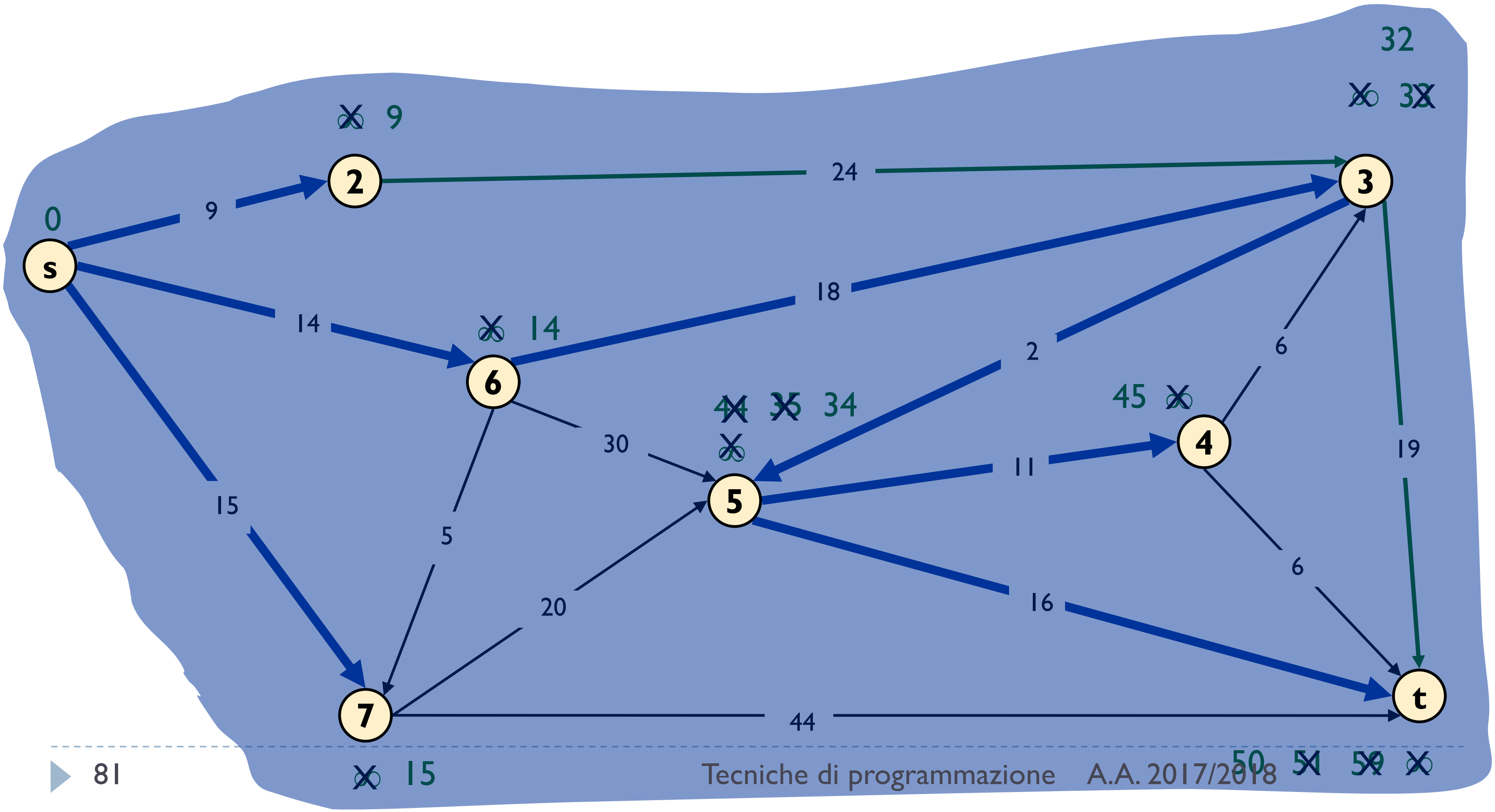
S = { s, 2, 3, 4, 5, 6, 7 }

Q = { t }



Dijkstra's Shortest Path Algorithm

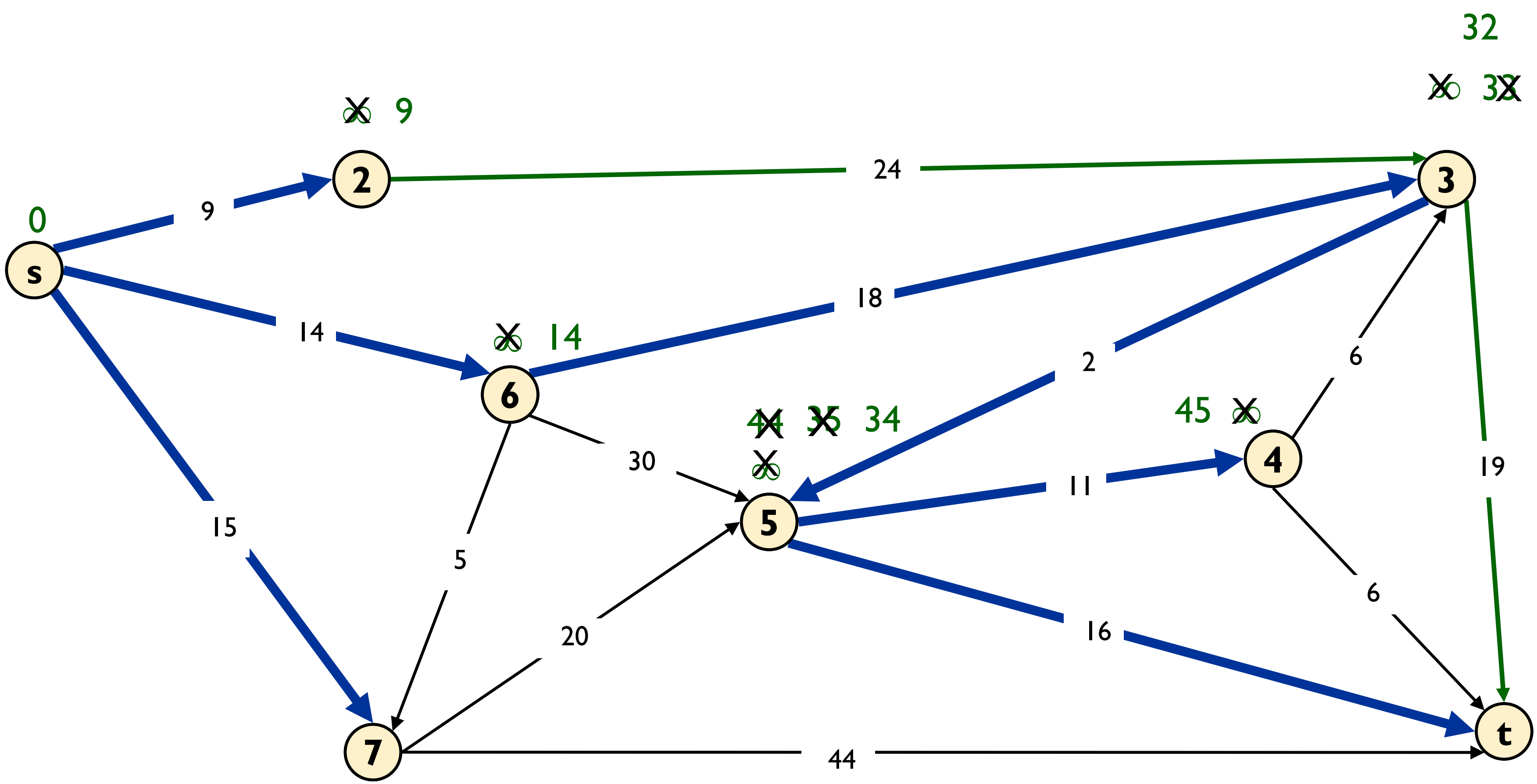
S = { s, 2, 3, 4, 5, 6, 7, t }
Q = { }



Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 4, 5, 6, 7, t }

Q = { }



Shortest Paths wrap-up

Algorithm	Problem	Efficiency	Limitation
Floyd-Warshall	AP	$O(V^3)$	No negative cycles
Bellman-Ford	SS	$O(V \cdot E)$	No negative cycles
Repeated Bellman-Ford	AP	$O(V^2 \cdot E)$	No negative cycles
Dijkstra	SS	$O(E + V \cdot \log V)$	No negative edges
Repeated Dijkstra	AP	$O(V \cdot E + V^2 \cdot \log V)$	No negative edges
Breadth-First visit	SS	$O(V + E)$	Unweighted graph

JGraphT

```
public class FloydWarshallShortestPaths<V,E>
public class BellmanFordShortestPath<V,E>
public class DijkstraShortestPath<V,E>
```

```
// APSP
List<GraphPath<V,E>>  getShortestPaths(V v)
GraphPath<V,E>        getShortestPath(V a, V b)

// SSSP
GraphPath<V,E>   getPath()
```

Resources

- Algorithms in a Nutshell, G. Heineman, G. Pollice, S. Selkow, O'Reilly, ISBN 978-0-596-51624-6, Chapter 6 http://shop.oreilly.com/product/9780596516246.do
- http://en.wikipedia.org/wiki/Floyd%E2%80%93Warshall_algorithm

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