POLITECNICO
DI TORINO

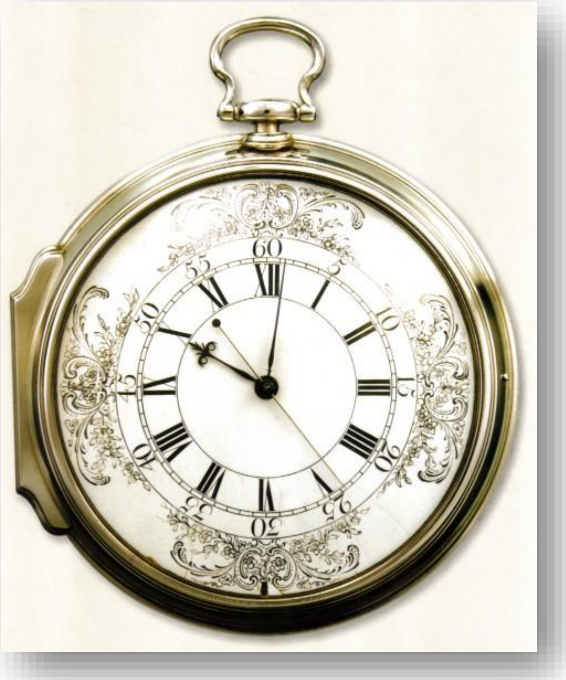
e-Lite

Computational complexity

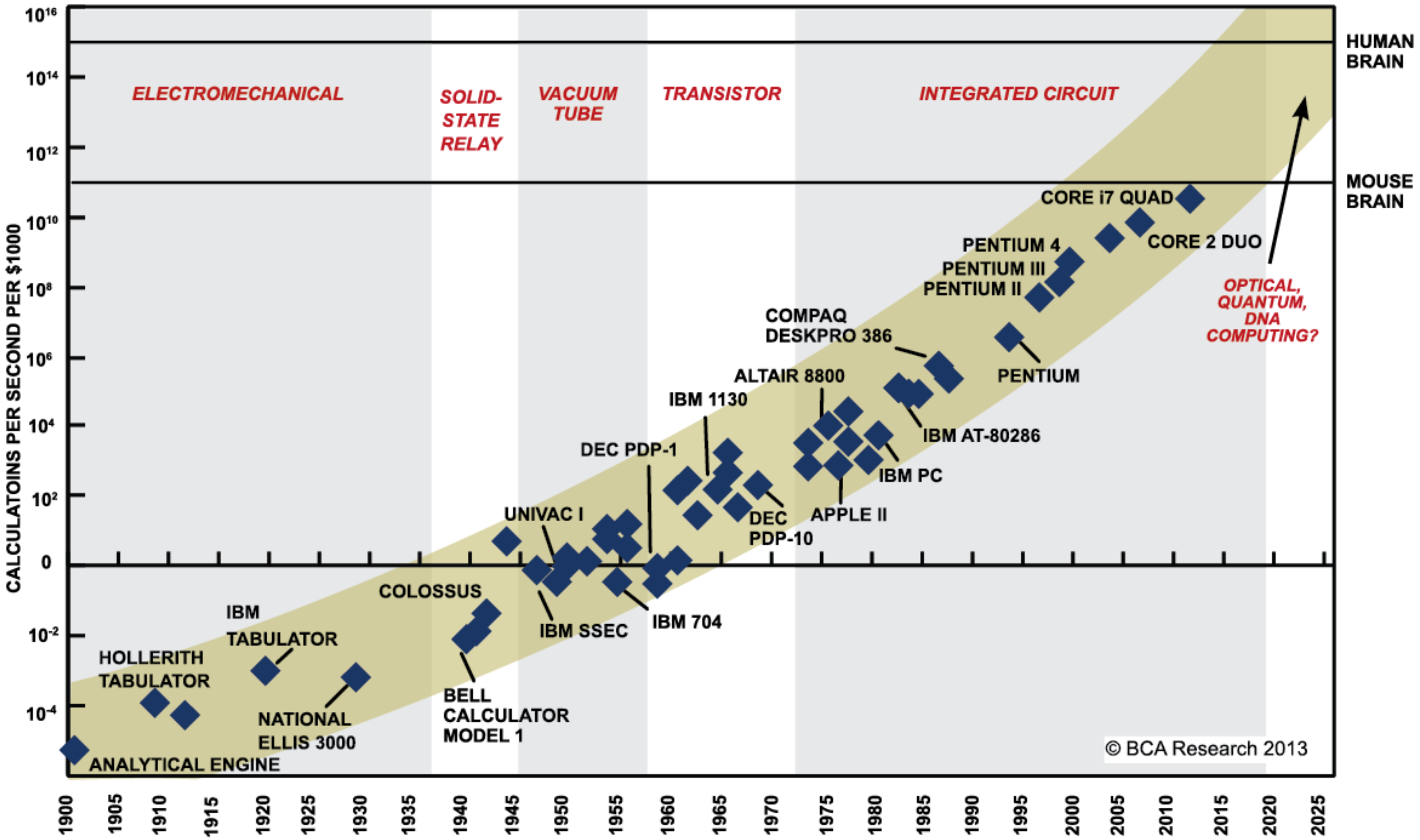
Tecniche di Programmazione – A.A. 2018/2019

How to Measure Efficiency?

- Critical resources
  - programmer's effort
  - time, space (disk, RAM)
- Analysis
  - empirical (run programs)
  - analytical (asymptotic algorithm analysis)
- Worst case vs. Average case

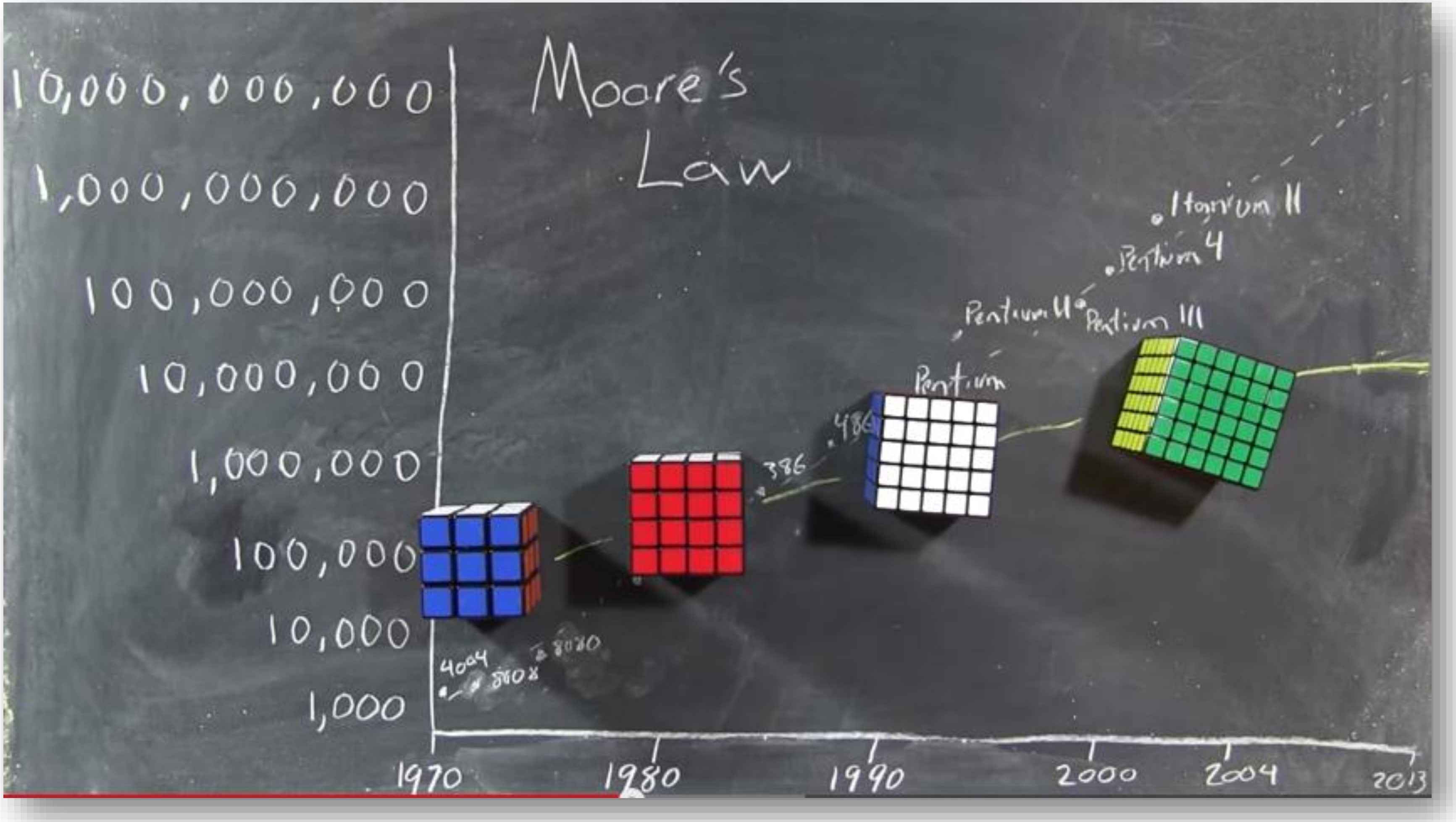


Moore's "Law"?



SOURCE: RAY KURZWEIL, "THE SINGULARITY IS NEAR: WHEN HUMANS TRANSCEND BIOLOGY", P.67, *THE VIKING PRESS*, 2006. DATAPOINTS BETWEEN 2000 AND 2012 REPRESENT BCA ESTIMATES.

Moore’s “Law”?


Moore's Law
10,000,000,000
1,000,000,000
100,000,000
10,000,000
1,000,000
100,000
10,000
1,000
4004
8008
8080
386
486
Pentium
Pentium II
Pentium III
Pentium 4
Itanium II
1970
1980
1990
2000
2004
2013

Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

			F	L			P	J	S				H	Y		R						T	Q	O
		X	M				T								J	P	U	B		C		S		
D	B	G	P		F	R						T			X				Q	Y	V	A	E	
	I		N				L	A	G	O		C				T			Y	B			R	
					K	Q	I	M			S		F			O	V				L	W		
		V						S		G						B			I	L		Y	K	D
									Q				Y	U					E	A		B		W
		P			M	A	N	R	K					F	S			Q		G				
	H		D	U	J	F	X		B	K			W								N	E		C
									P		R		M	T	D	C		L	U	I		J		
		N		K		H								P		M	C		O	R			G	Q
Q		B				V				X	I	J					S		K			M	A	T
	U	D						W	C		L	G	K	A	Q	Y			H				P	
			X	I	A		S		N	H				O		U					B	F	C	
	G	J		W	L	U	Q									V		R		E		I	X	
		M		N						I		D	Q		K		G	S	P	U	F			
B				H		P	D				F	Y	A	L	I				M					
	A				Y	C			J		U			G	F									
			I					N		W	O	V		B					T		S	D		
			C		V		R		L		T	P	N					O	A	M	I		Y	K
	T			O	I					N	J		C	R				V						M
Y	N	U				B						Q		X			W				P	C	O	
		W		M	U		C		V	B	P	I					H		F	D	K	Q		
	C		G					T		E				M			O		L				V	X
K	X			V	R	J			F			H					Q	U			T		B	

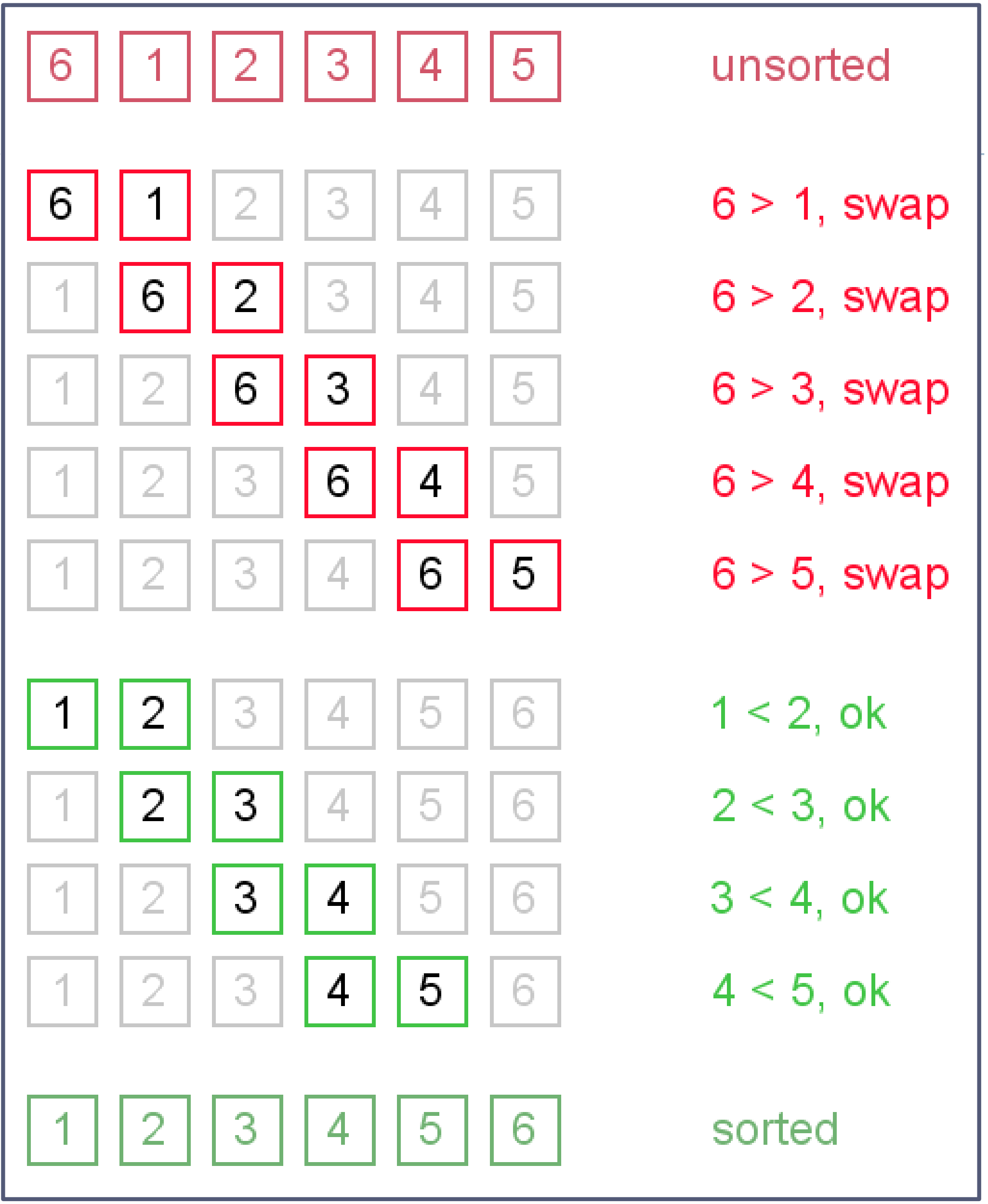
Problems and Algorithms

- We know the efficiency of the solution
- … but what about the difficulty of the problem?
- Different concepts
  - Algorithm complexity
  - Problem complexity

Analytical Approach

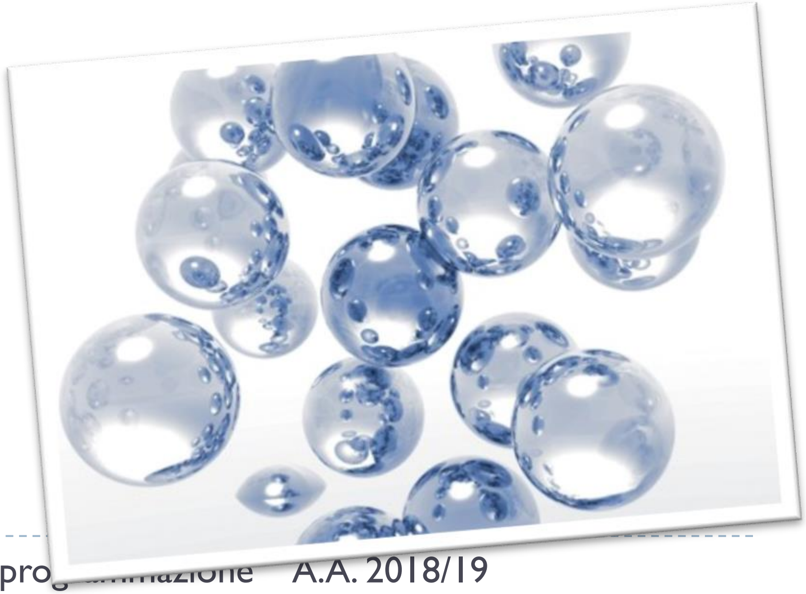
- An algorithm is a mapping
- For most algorithms, running time depends on “size” of the input
- Running time is expressed as T(n)
  - some function T
  - input *size* n

Bubble sort


6 1 2 3 4 5 unsorted
6 1 2 3 4 5 6 > 1, swap
1 6 2 3 4 5 6 > 2, swap
1 2 6 3 4 5 6 > 3, swap
1 2 3 6 4 5 6 > 4, swap
1 2 3 4 6 5 6 > 5, swap
1 2 3 4 5 6 1 < 2, ok
1 2 3 4 5 6 2 < 3, ok
1 2 3 4 5 6 3 < 4, ok
1 2 3 4 5 6 4 < 5, ok
1 2 3 4 5 6 sorted

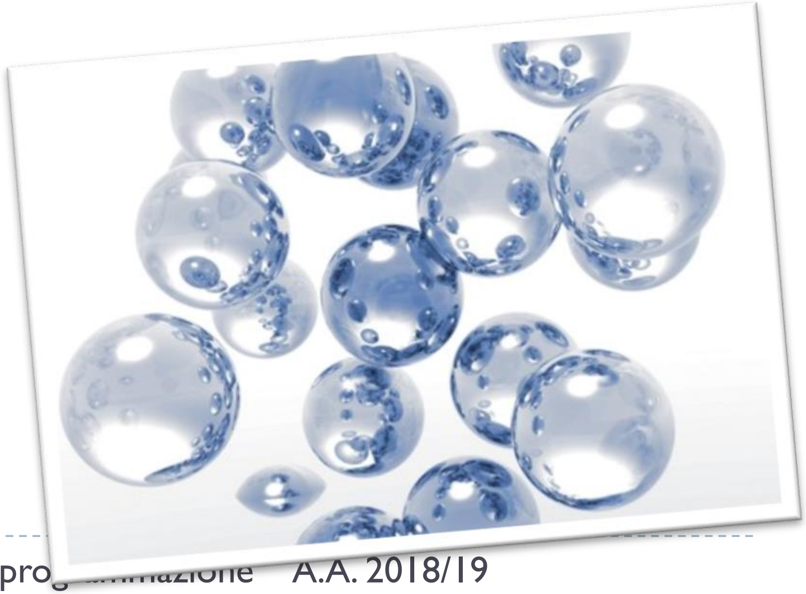
Analysis

- The bubble sort takes $(n^2-n)/2$ “steps”
- Different implementations/assembly languages
  - Program A on an Intel Pentium IV: $T(n) = 58*(n^2-n)/2$
  - Program B on a Motorola: $T(n) = 84*(n^2-2n)/2$
  - Program C on an Intel Pentium V: $T(n) = 44*(n^2-n)/2$
- Note that each has an n^2 term
  - as n increases, the other terms will drop out



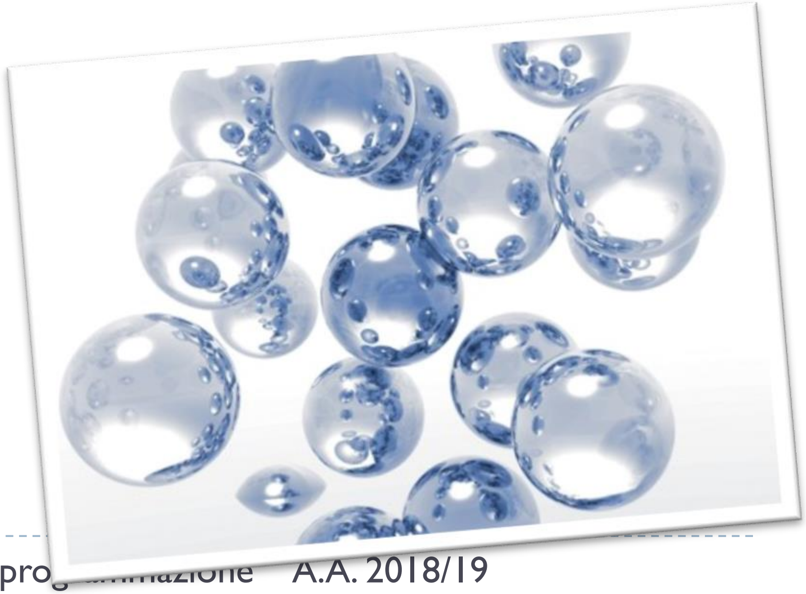
Analysis

- As a result:
  - Program A on Intel Pentium IV: $T(n) \approx 29n^2$
  - Program B on Motorola: $T(n) \approx 42n^2$
  - Program C on Intel Pentium V: $T(n) \approx 22n^2$



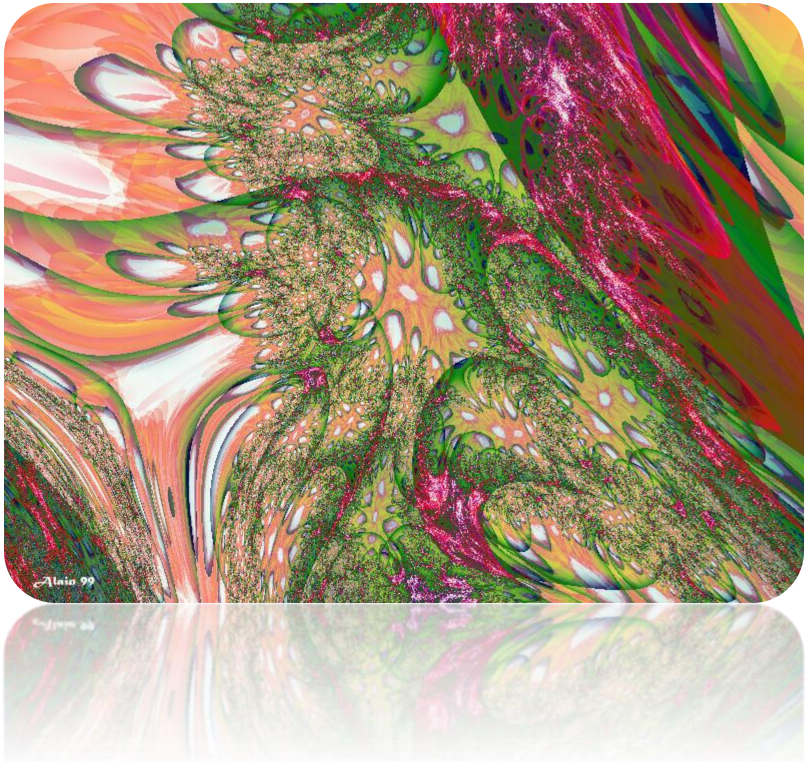
Analysis

- As processors change, the constants will always change
  - The exponent on n will not
  - We should not care about the constants
- As a result:
  - Program A: $T(n) \approx n^2$
  - Program B: $T(n) \approx n^2$
  - Program C: $T(n) \approx n^2$
- Bubble sort: $T(n) \approx n^2$



Complexity Analysis

- $O(\cdot)$
  - big o (big oh)
- $\Omega(\cdot)$
  - big omega
- $\Theta(\cdot)$
  - big theta

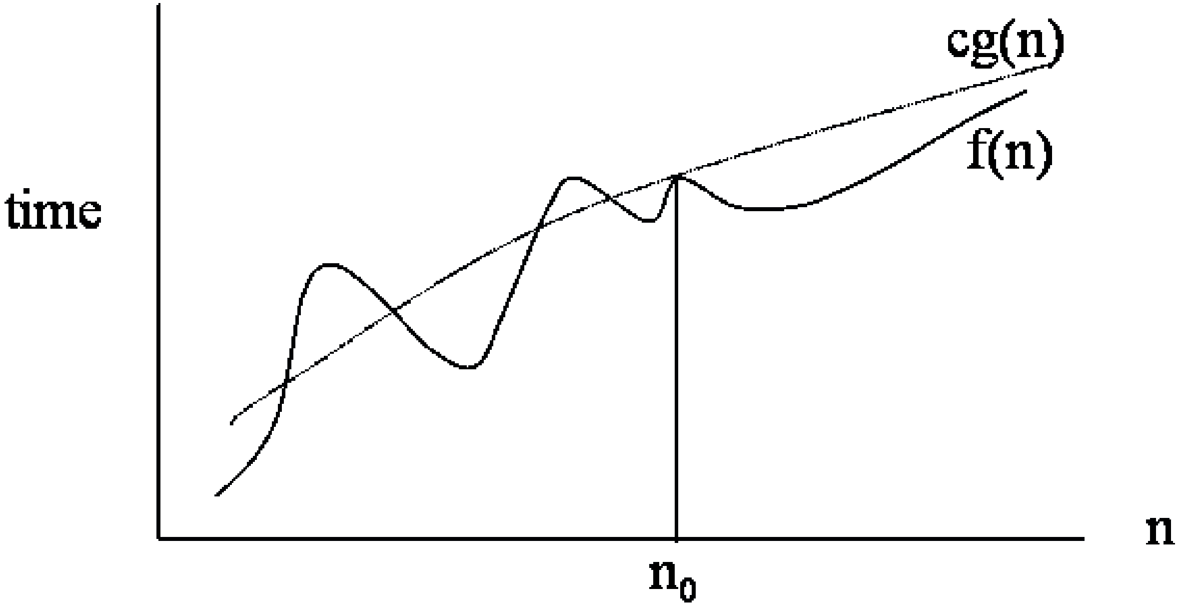


O(·)

- Upper Bounding Running Time

Upper Bounding Running Time

- f(n) is O(g(n)) if f grows "at most as fast as" g



Big-O (formal)

- Let *f* and *g* be two functions such that

$$f(n): N \to R^{+} \ and \ g(n): N \to R^{+}$$

- if there exists positive constants c and n_0 such that

$$f(n) \leq cg(n), \quad for\ all\ n > n_0$$

- then we can write

$$f(n) = O(g(n))$$

Big-O (formal alt)

- Let *f* and *g* be two functions such that

$$f(n): N \rightarrow R^{+} \ and \ g(n): N \rightarrow R^{+}$$

- if there exists positive constants c and n_0 such that

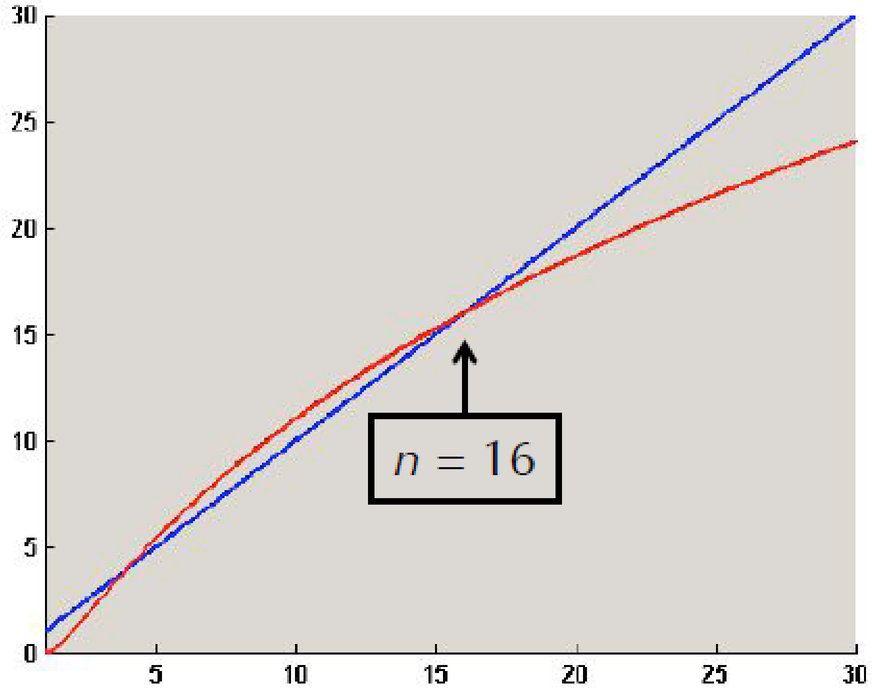
$$0 \leq \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = c < \infty$$

- then we can write

$$f(n) = O(g(n))$$

Example

- $(\log n)^2 = O(n)$



$f(n) = (\log n)^2$

$g(n) = n$

$(\log n)^2 \leq n$ for all $n \geq 16$, so $(\log n)^2$ is $O(n)$

Notational Issues

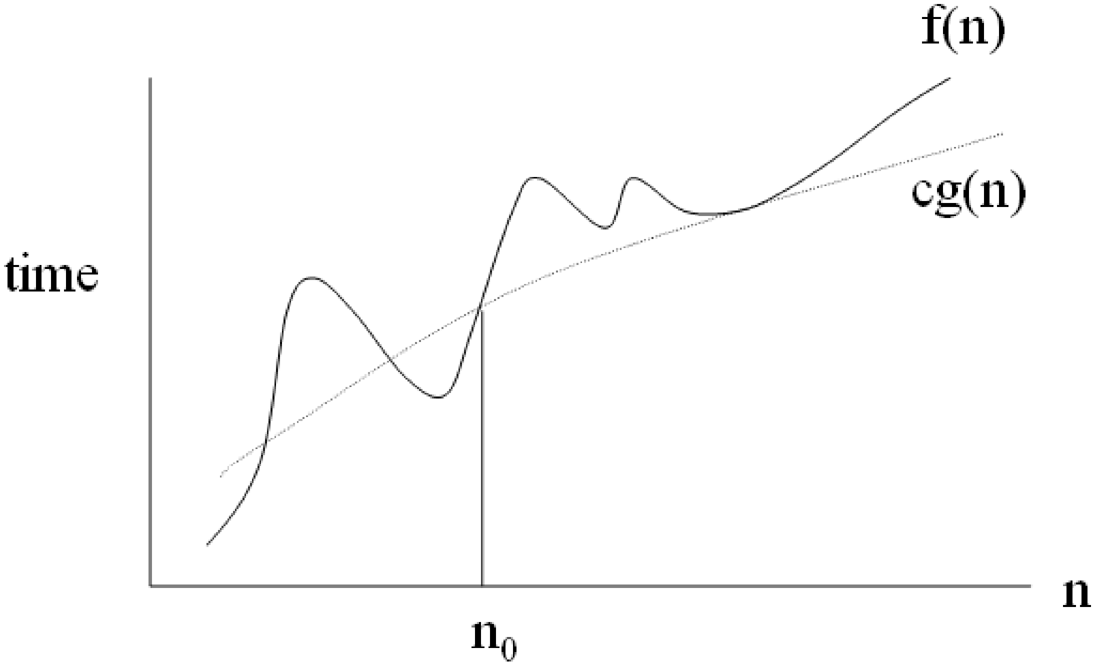
- Big-O notation is a way of comparing functions
- Notation quite unconventional
  - e.g., $3x^3 + 5x^2 – 9 = O(x^3)$
- Doesn't mean
- "$3x^3 + 5x^2 – 9$ equals the function $O(x^3)$"
- "$3x^3 + 5x^2 – 9$ is big oh of x^3"
- But
  - "$3x^3+5x^2 –9$ is dominated by x^3"

Common Misunderstanding

- $3x^3 + 5x^2 – 9 = O(x^3)$
- However, also true are:
  - $3x^3 + 5x^2 – 9 = O(x^4)$
  - $x^3 = O(3x^3 + 5x^2 – 9)$
  - $\sin(x) = O(x^4)$
- Note:
  - Usage of big-O typically involves mentioning only the most dominant term
  - "The running time is $O(x^{2.5})$"

Lower Bounding Running Time

- f(n) is Ω(g(n)) if f grows “at least as fast as” g



- **cg(n) is an approximation to f(n) bounding from below**

Big-Omega (formal)

- Let *f* and *g* be two functions such that

$$f(n): N \rightarrow R^{+} \; and \;\; g(n): N \rightarrow R^{+}$$

- if there exists positive constants c and n_0 such that

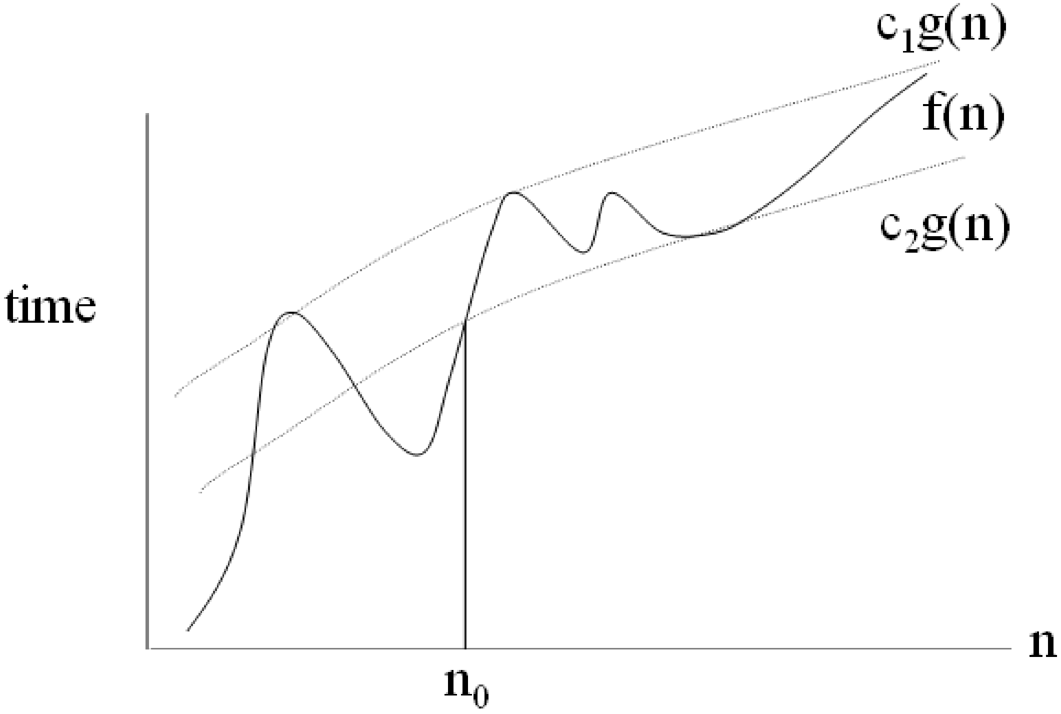
$$f(n) \geq cg(n), \quad for\ all\ n > n_0$$

- then we can write

$$f(n) = \Omega(g(n))$$

Tightly Bounding Running Time

- f(n) is Θ(g(n)) if f is essentially the same as g, to within a constant multiple



Big-Theta (formal)

- Let *f* and *g* be two functions such that

$$f(n): N \rightarrow R^+ \ and \ g(n): N \rightarrow R^+$$

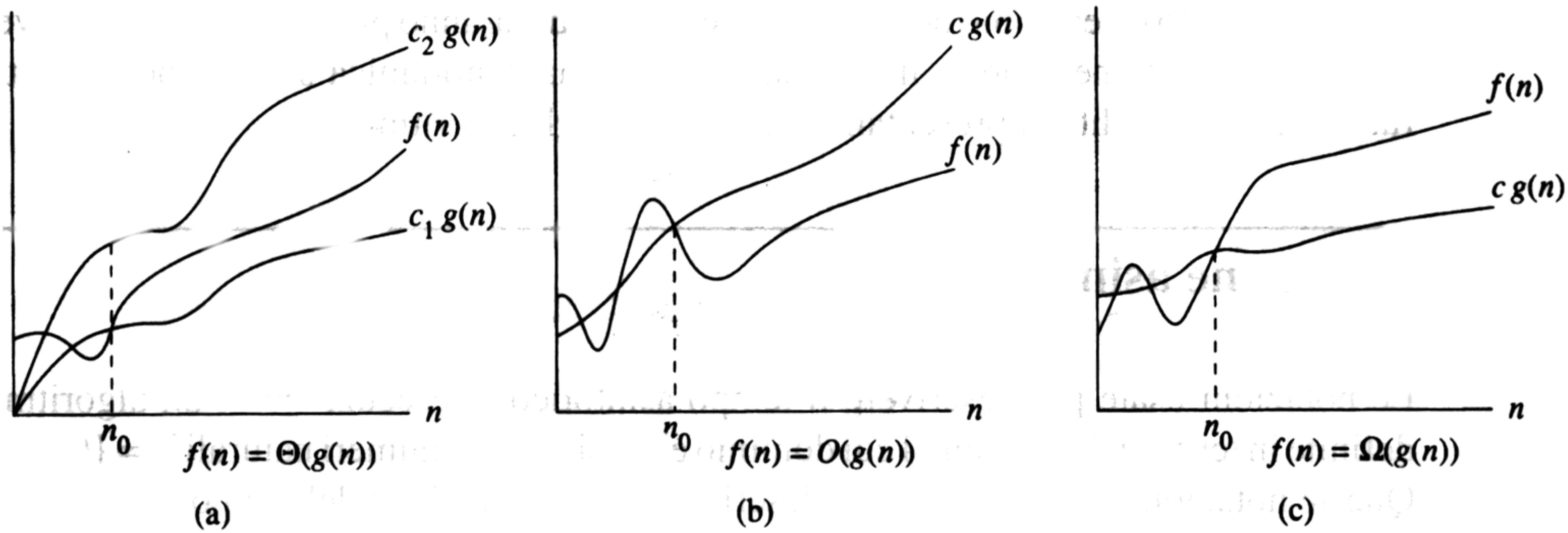
- if there exists positive constants c_1, c_2 and n_0 such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n), \quad for\ all\ n > n_0$$

- then we can write

$$f(n) = \Theta(g(n))$$

Big-Θ, Big-O, and Big-Ω



$f(n) = \Theta(g(n))$ $f(n) = O(g(n))$ $f(n) = \Omega(g(n))$

(a) (b) (c)

Big-Ω and Big-O

- Big-Ω: reverse of big-O. I.e.

$$f(x) = \Omega(g(x))$$

iff

$$g(x) = O(f(x))$$

- so f(x) asymptotically dominates g(x)

Big-Θ = Big-O and Big-Ω

- Big-Θ: domination in both directions. I.e.

$$f(x) = \Theta(g(x))$$

iff

$$f(x) = O(g(x)) \;\&\&\; f(x) = \Omega(g(x))$$

Problem

- Order the following from smallest to largest asymptotically. Group together all functions which are big-Θ of each other:

$$x+\sin x, \ln x, x+\sqrt{x}, \frac{1}{x}, 13+\frac{1}{x}, 13+x, e^x, x^e, x^x$$

$$(x+\sin x)(x^{20}-102), x\ln x, x(\ln x)^2, \lg_2 x$$

Solution

$1/x$

$13+1/x$

$\ln x \ \lg_2 x$

$x+\sin x,$

$x \ln x$

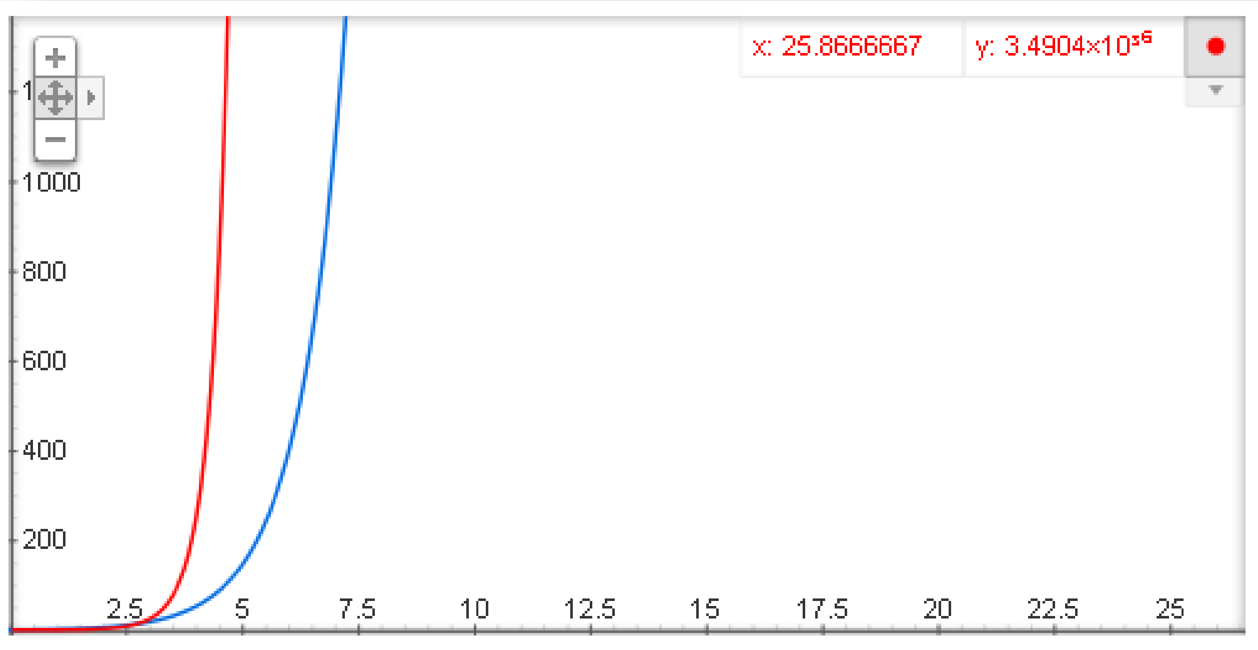
$x(\ln x)^2$

x^e

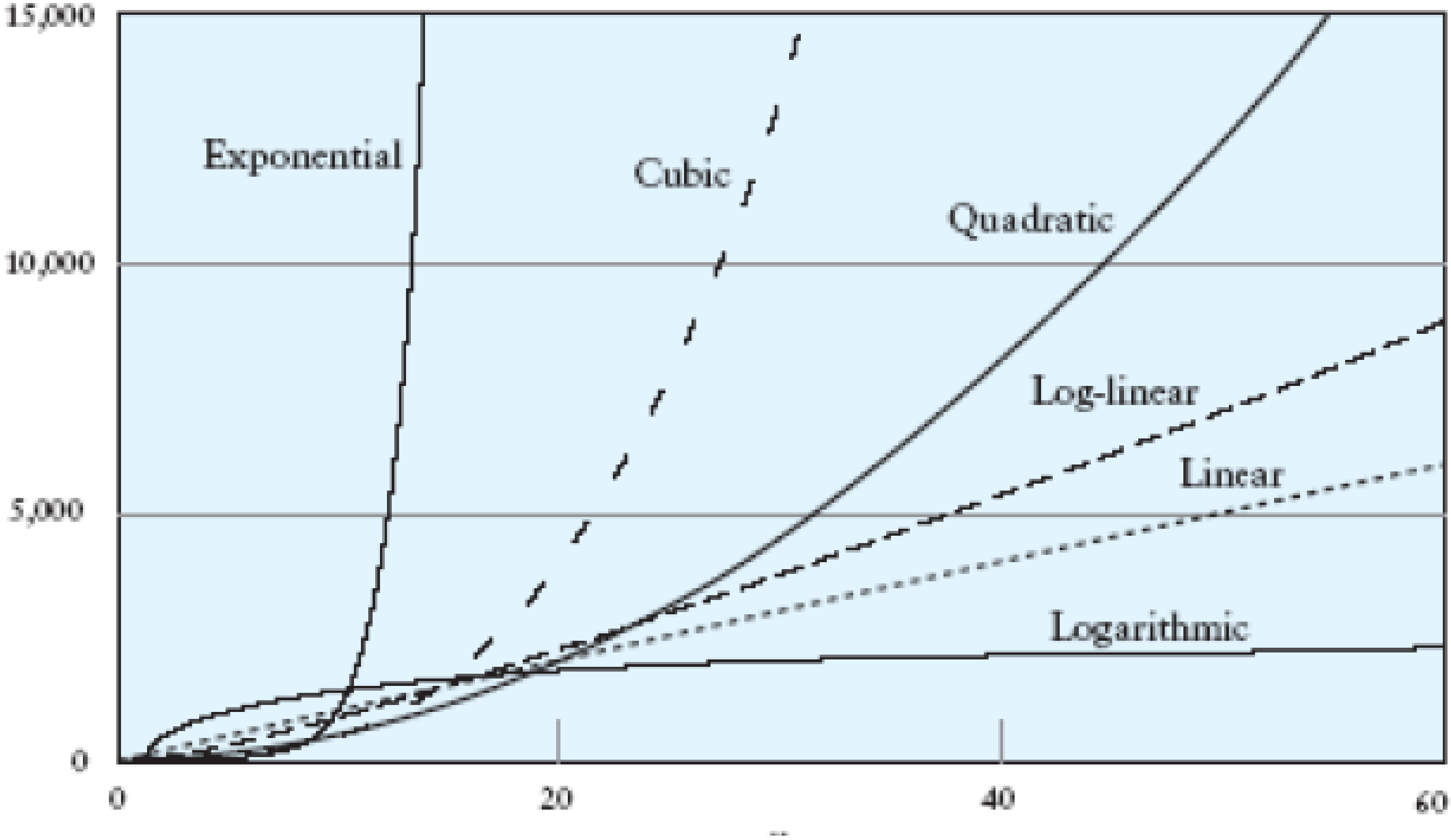
$(x+\sin x)(x^{20}-102)$

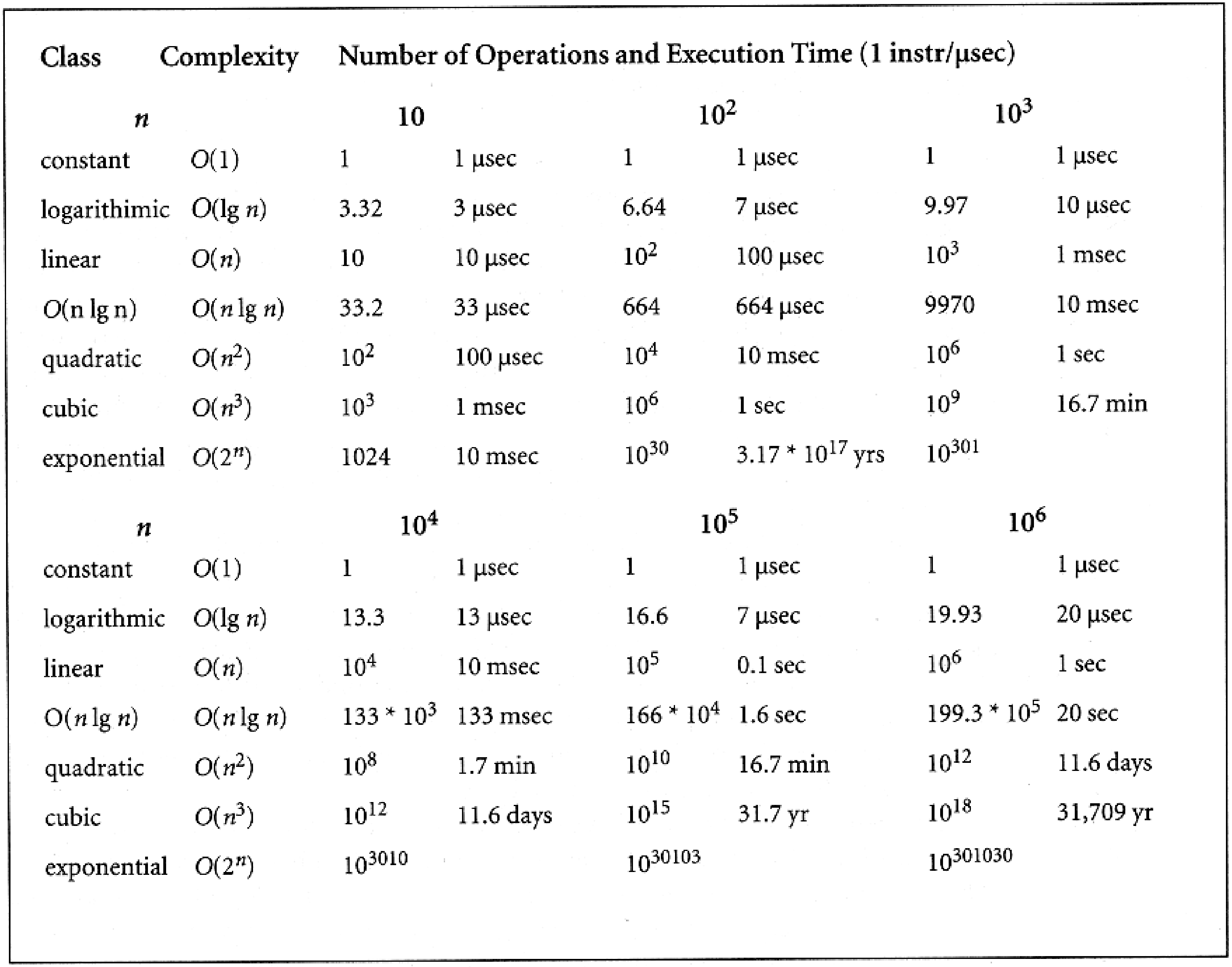
e^x

x^x



Practical approach

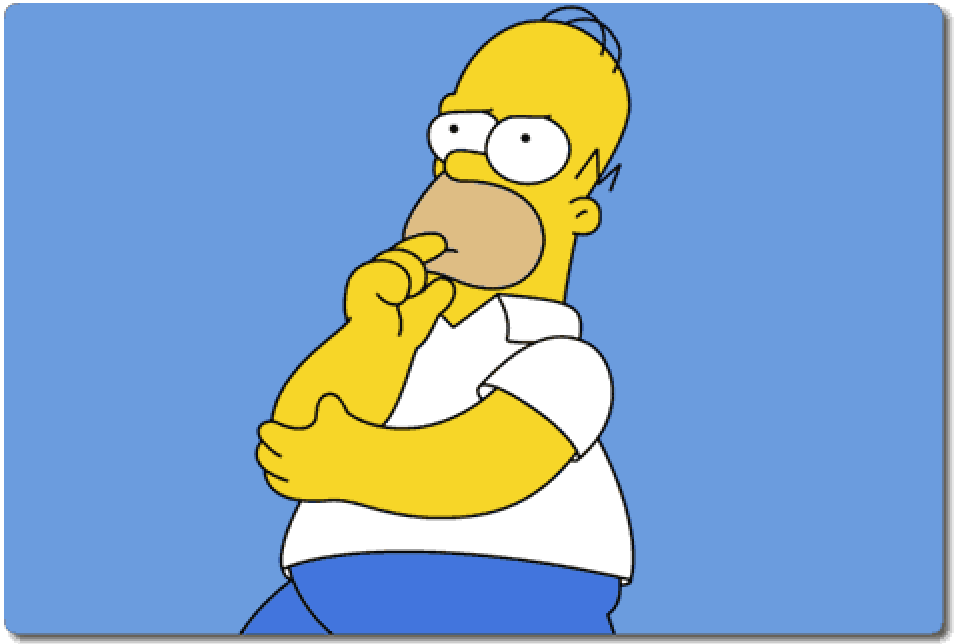




Class	Complexity	Number of Operations and Execution Time (1 instr/µsec)					
n		10		10^2		10^3	
constant	$O(1)$	1	1 µsec	1	1 µsec	1	1 µsec
logarithimic	$O(\lg n)$	3.32	3 µsec	6.64	7 µsec	9.97	10 µsec
linear	$O(n)$	10	10 µsec	10^2	100 µsec	10^3	1 msec
O(n lg n)	$O(n \lg n)$	33.2	33 µsec	664	664 µsec	9970	10 msec
quadratic	$O(n^2)$	10^2	100 µsec	10^4	10 msec	10^6	1 sec
cubic	$O(n^3)$	10^3	1 msec	10^6	1 sec	10^9	16.7 min
exponential	$O(2^n)$	1024	10 msec	10^{30}	$3.17 * 10^{17}$ yrs	10^{301}	
n		10^4		10^5		10^6	
constant	$O(1)$	1	1 µsec	1	1 µsec	1	1 µsec
logarithmic	$O(\lg n)$	13.3	13 µsec	16.6	7 µsec	19.93	20 µsec
linear	$O(n)$	10^4	10 msec	10^5	0.1 sec	10^6	1 sec
$O(n \lg n)$	$O(n \lg n)$	$133 * 10^3$	133 msec	$166 * 10^4$	1.6 sec	$199.3 * 10^5$	20 sec
quadratic	$O(n^2)$	10^8	1.7 min	10^{10}	16.7 min	10^{12}	11.6 days
cubic	$O(n^3)$	10^{12}	11.6 days	10^{15}	31.7 yr	10^{18}	31,709 yr
exponential	$O(2^n)$	10^{3010}		10^{30103}		10^{301030}	

Would it be possible?

Algorithm	Foo	Bar
Complexity	$O(n^2)$	$O(2^n)$
n = 100	10s	4s
n = 1000	12s	4.5s



Determination of Time Complexity

- Because of the approximations available through Big-O , the actual T(n) of an algorithm is not calculated
  - T(n) may be determined empirically
- Big-O is usually determined by application of some simple 5 rules

Rule #1

- **Simple** program **statements** are assumed to take a constant amount of time which is

  O(1)

Rule #2

- Differences in execution time of simple statements is ignored

Rule #3

- In **conditional** statements the worst case is always used

Rule #4 – the “sum” rule

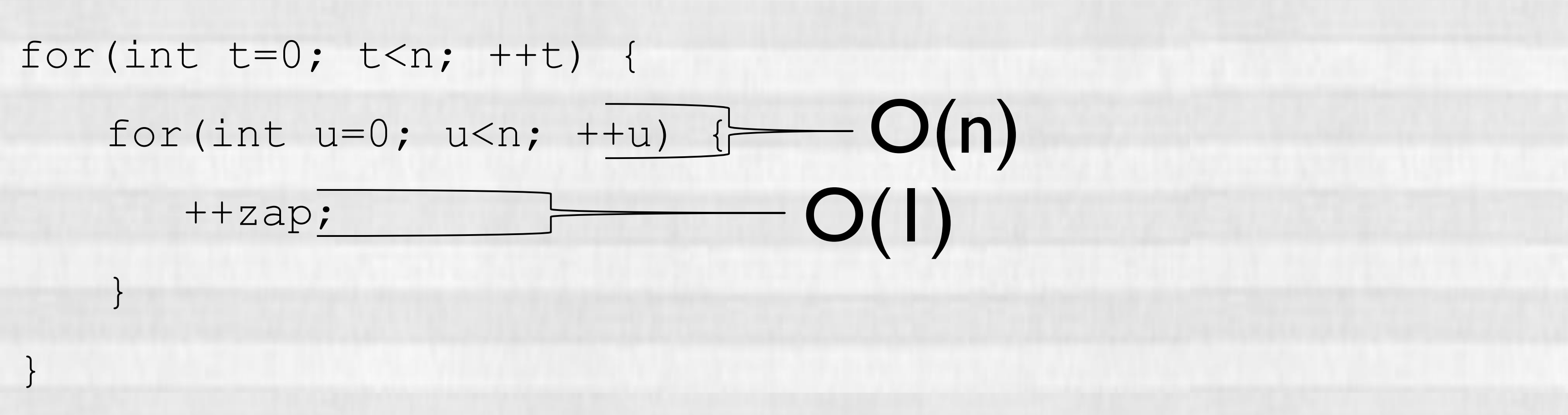
- The running time of a **sequence** of steps has the order of the running time of the largest
- E.g.,
  - $f(n) = O(n^2)$
  - $g(n) = O(n^3)$
  - $f(n) + g(n) = O(n^3)$

Worst case
(valid for big-O, not
for big-θ)

Rule #5 – the "product" rule

- If two processes are constructed such that the second process is **repeated** a number of times for each execution of the first process, then O is equal to the **product** of the orders of magnitude of the two processes
- E.g.,
  - For example, a two-dimensional array has one for loop inside another and each internal loop is executed n times for each value of the external loop.
  - The function is O(n^2)

Nested Loops



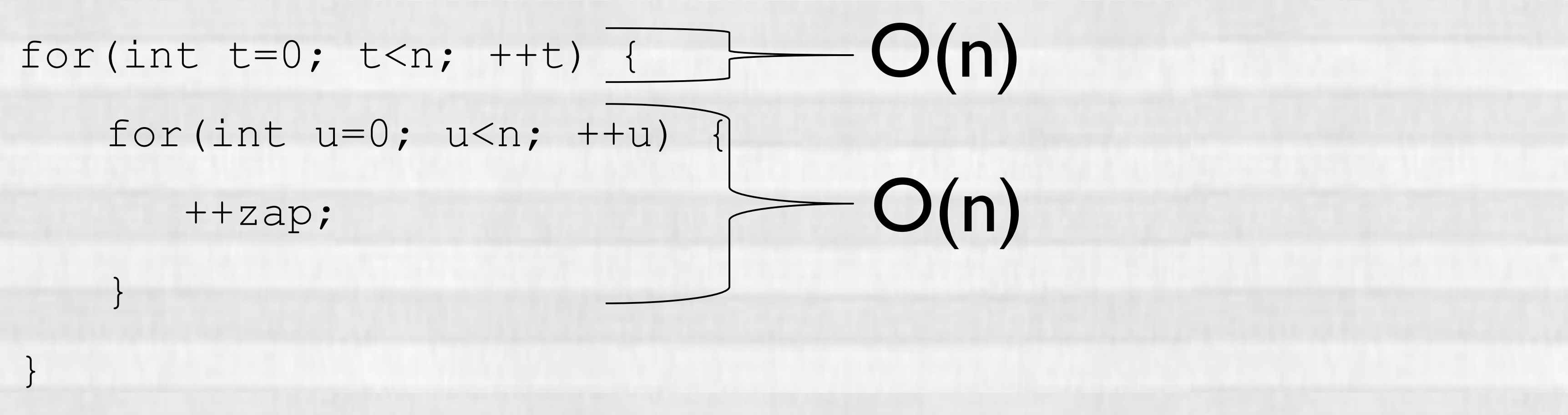
```
for(int t=0; t<n; ++t) {
   for(int u=0; u<n; ++u) {
      ++zap;
   }
}
```

Nested Loops

```
for(int t=0; t<n; ++t) {
    for(int u=0; u<n; ++u) {
        ++zap;
    }
}
```

O(n*1)

Nested Loops



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for(int t=0; t<n; ++t) {
    for(int u=0; u<n; ++u) {
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    }
}
```

Nested Loops

```
for(int t=0; t<n; ++t) {
   for(int u=0; u<n; ++u) {
      ++zap;
   }
}
```

O(n^2)

Nested Loops

- Note: Running time grows with nesting rather than the length of the code

```
for(int t=0; t<n; ++t) {
    for(int u=0; u<n; ++u) {
        ++zap;
    }
}
```

$O(n^2)$

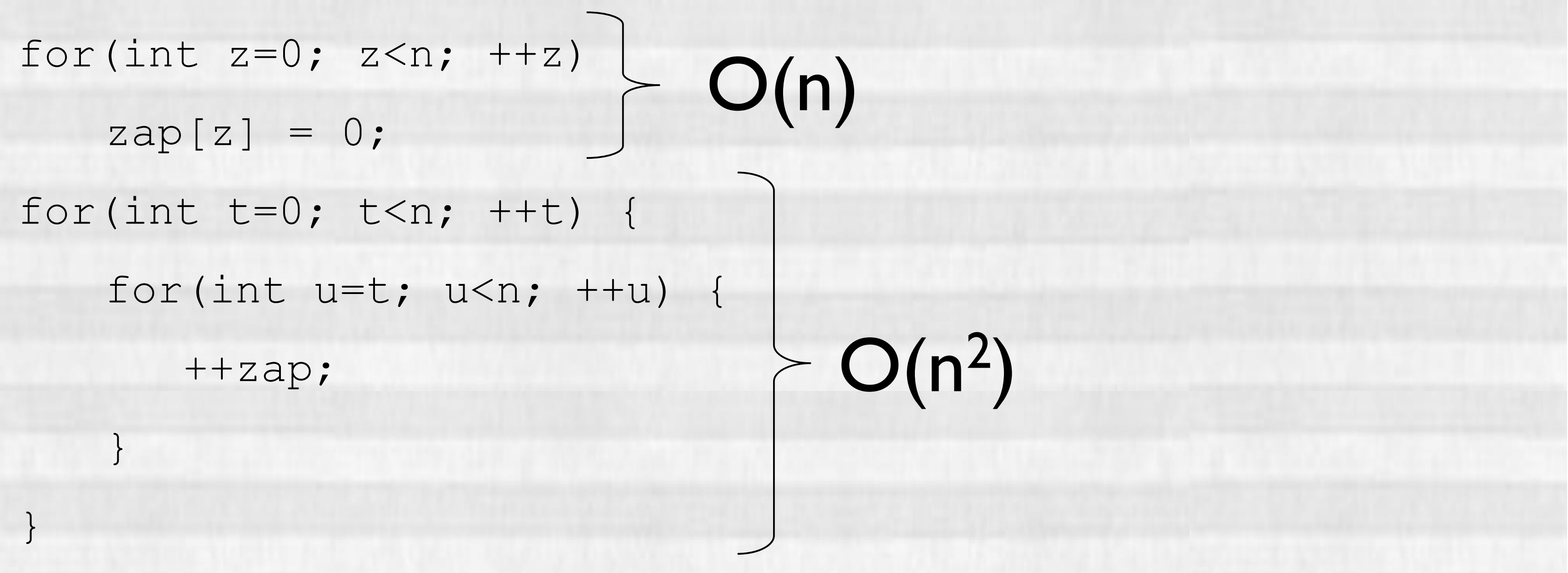
More Nested Loops

```
for(int t=0; t<n; ++t) {
    for(int u=t; u<n; ++u) {
        ++zap;
    }
}
```

$$n-t$$

$$\boxed{\sum_{i=0}^{n-1}(n-i)=\frac{n(n-1)}{2}=\frac{n^2-n}{2}=O(n^2)}$$

Sequential statements



```
for(int z=0; z<n; ++z)
   zap[z] = 0;
for(int t=0; t<n; ++t) {
   for(int u=t; u<n; ++u) {
      ++zap;
   }
}
```

- Running time: $\max(O(n), O(n^2)) = O(n^2)$

Conditionals

```
for(int t=0; t<n; ++t) {
    if(t%2) {
        for(int u=t; u<n; ++u) {
            ++zap;
        }
    } else {
        zap = 0;
    }
}
```

O(n)

O(1)

Conditionals

```
for(int t=0; t<n; ++t) {
    if(t%2) {
        for(int u=t; u<n; ++u) {
            ++zap;
        }
    } else {
        zap = 0;
    }
}
```

$O(n^2)$

Tips

- Focus only on the dominant (high cost) operations and avoid a line-by-line exact analysis
- Break algorithm down into “known” pieces
- Identify relationships between pieces
  - Sequential is additive
  - Nested (loop / recursion) is multiplicative
- Drop constants
- Keep only dominant factor for each variable

Caveats

- Real time vs. complexity

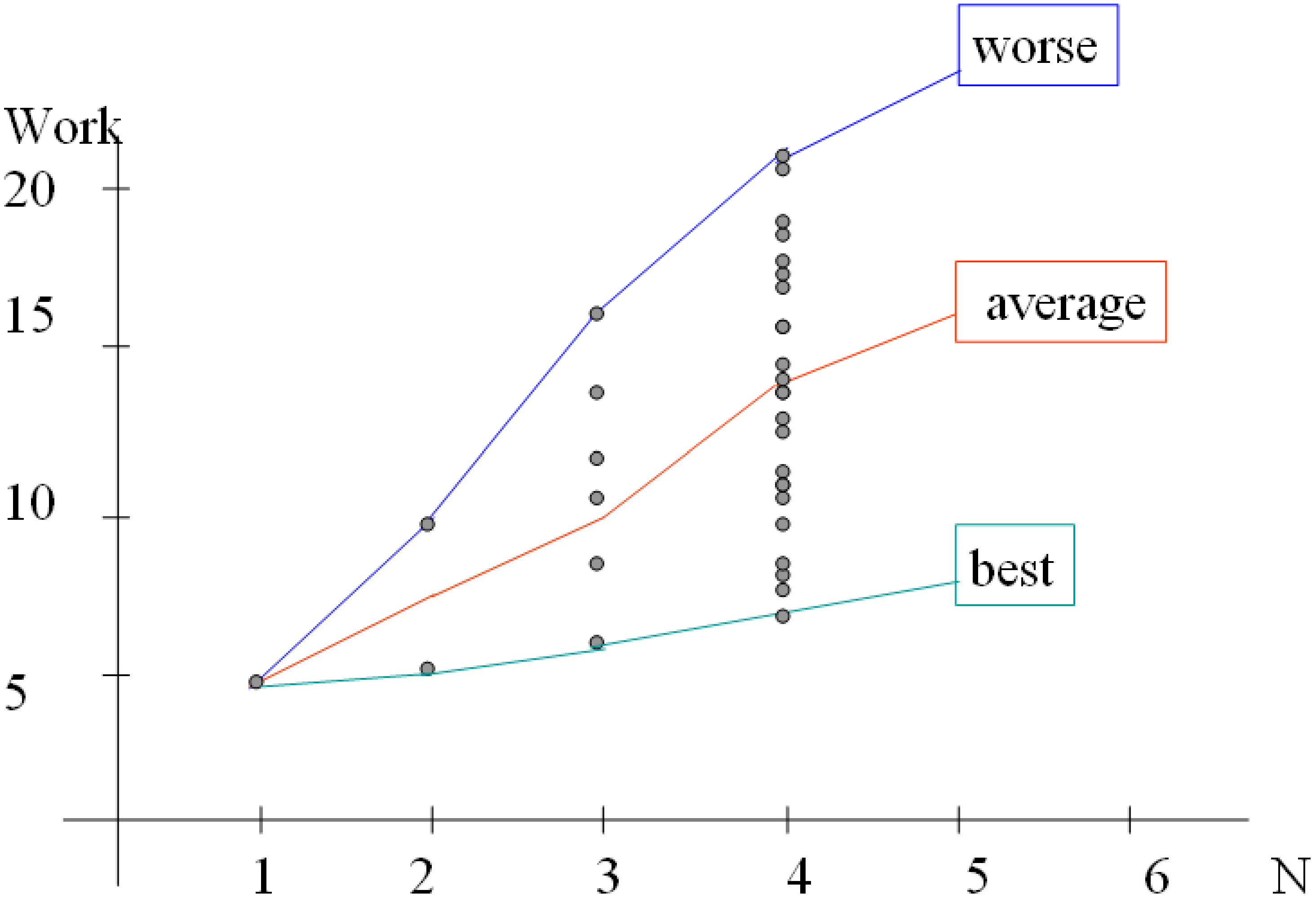
Caveats

- Real time vs. complexity
- CPU time vs. RAM vs. disk

Caveats

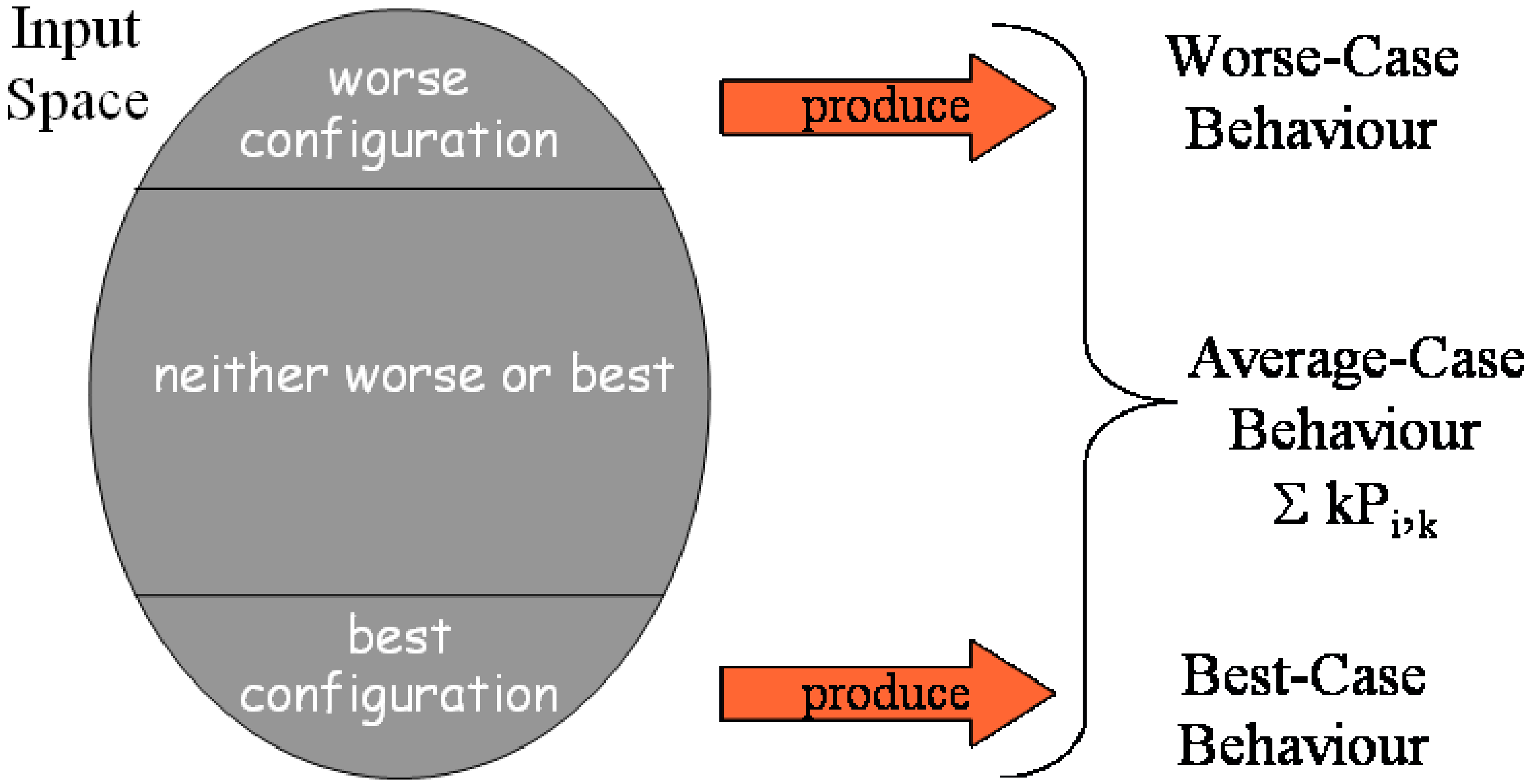
- Real time vs. complexity
- CPU time vs. RAM vs. disk
- Worse, Average or Best Case?

Worse, Average or Best Case?



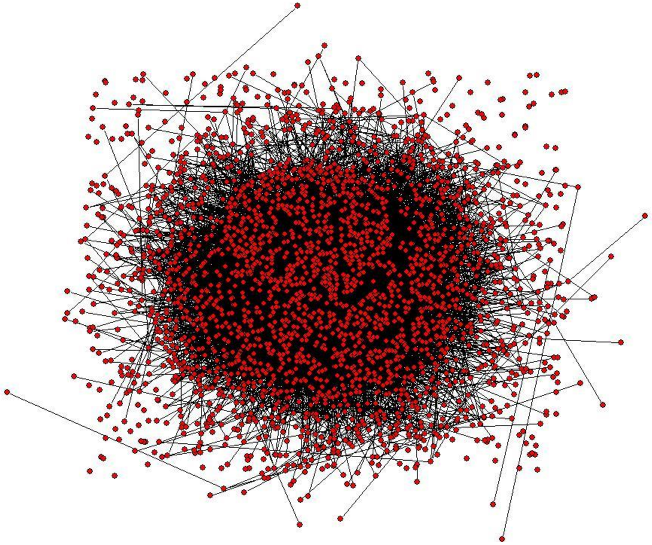
Worse, Average or Best Case?

- Depends on input problem instance type



Computational Complexity Theory

- In computer science, computational complexity theory is the branch of the theory of computation that studies the resources, or cost, of the computation required to solve a given computational problem
- Complexity theory analyzes the difficulty of computational problems in terms of many different computational resources



Note

Solve a problem

vs.

Verify a solution

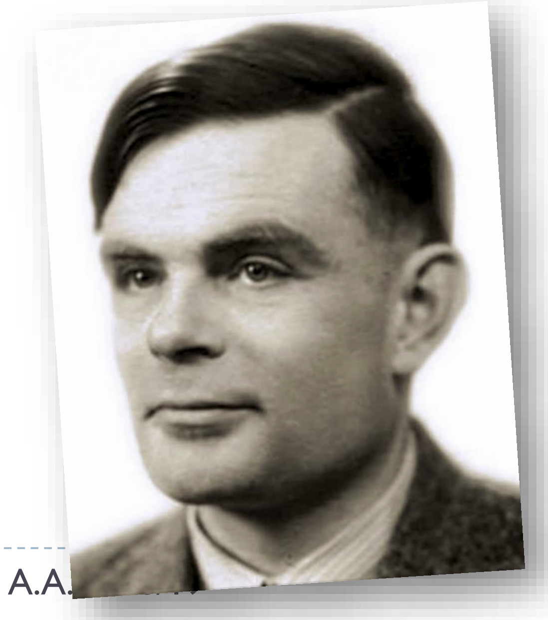
- E.g.,
  - Sort
  - Shortest path

Complexity Classes

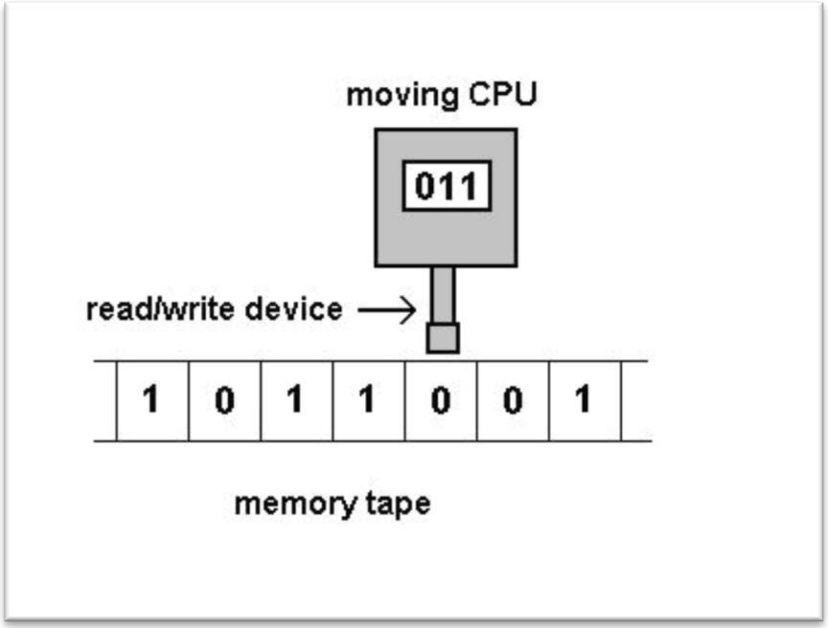
- A complexity class is the set of all of the computational problems which can be solved using a certain amount of a certain computational resource

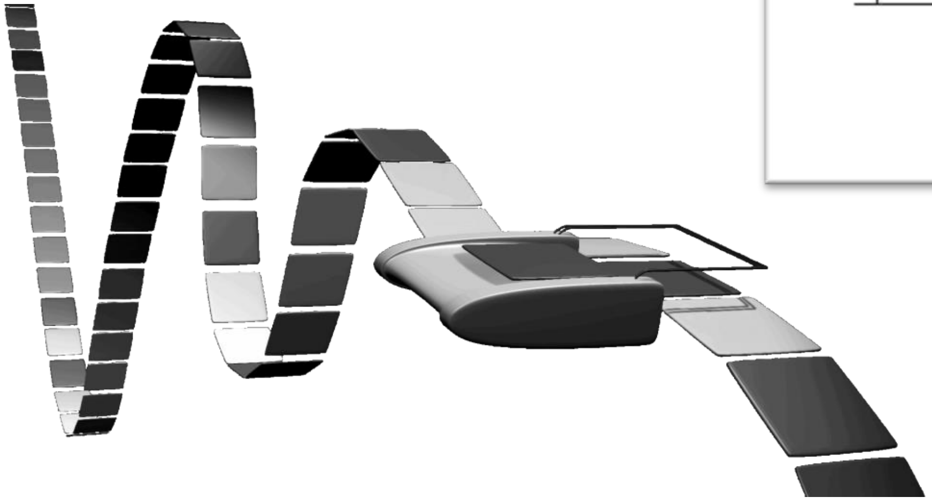
Deterministic Turing Machine

- Deterministic or Turing machines are extremely basic symbol-manipulating devices which — despite their simplicity — can be adapted to simulate the logic of any computer that could possibly be constructed
- Described in 1936 by Alan Turing.
  - Not meant to be a practical computing technology
  - Technically feasible
  - A thought experiment about the limits of mechanical computation



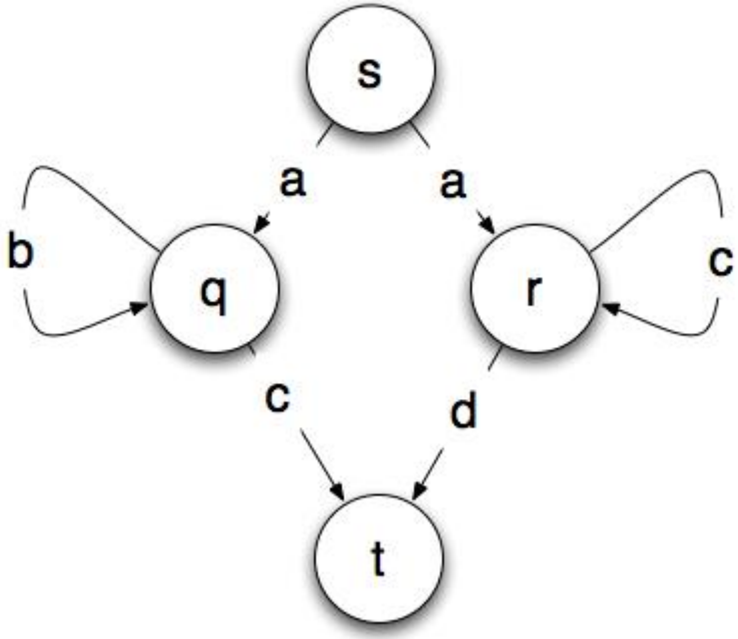
Deterministic Turing Machine

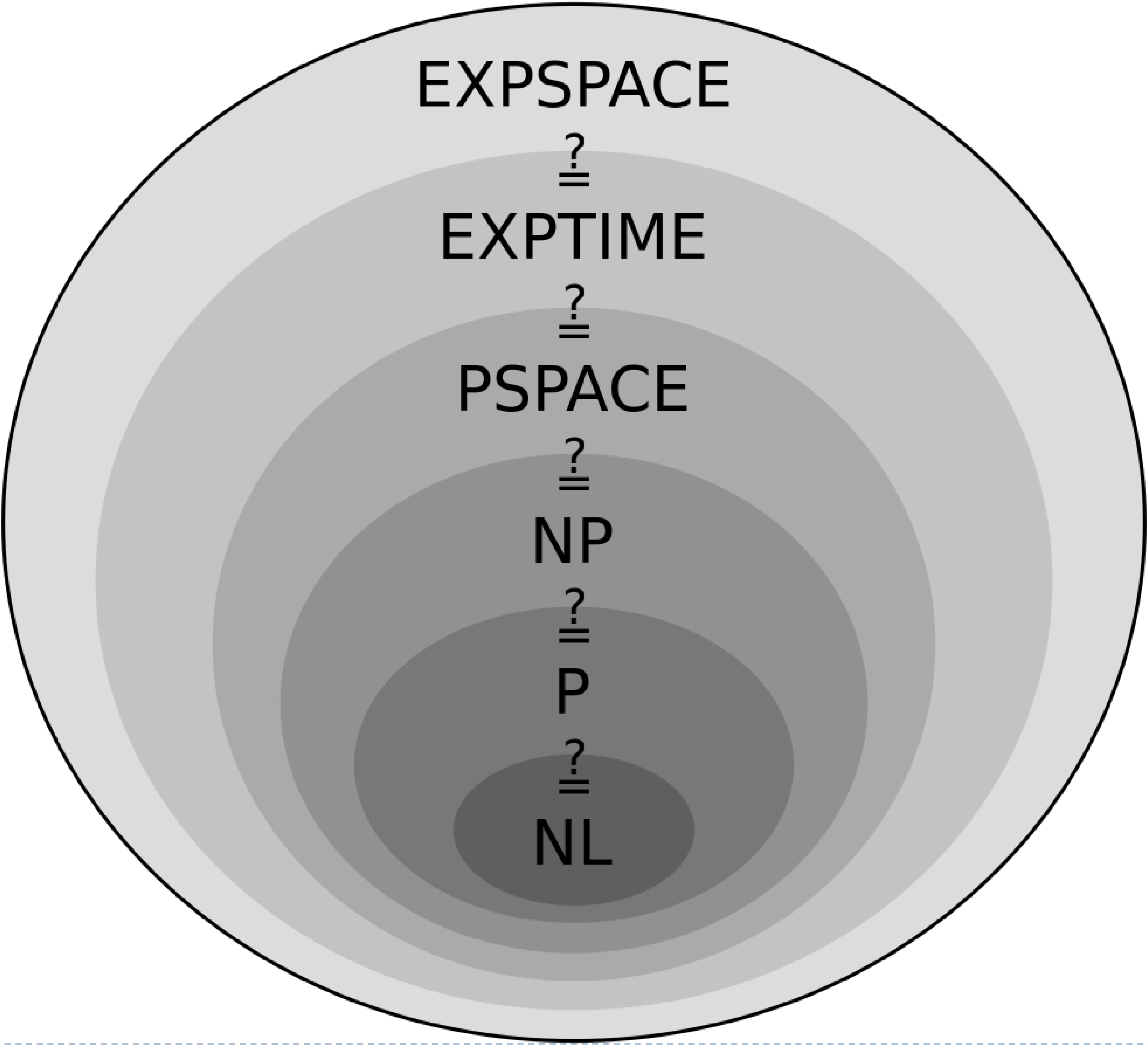




Non-Deterministic Turing Machine

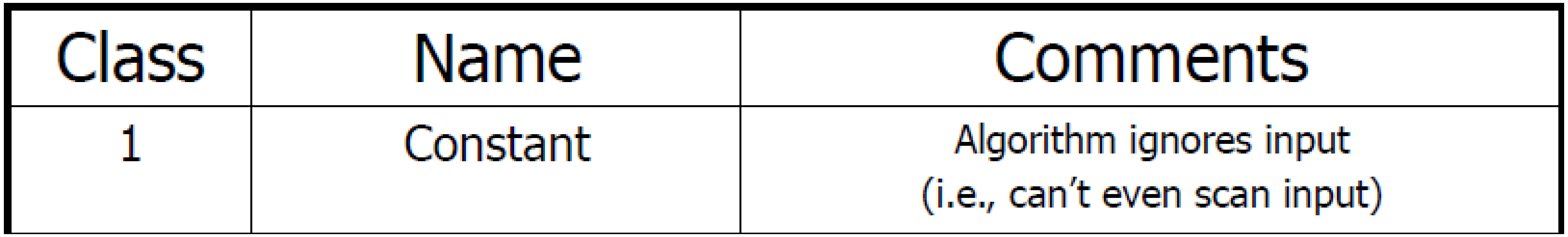
- Turing machine whose control mechanism works like a non-deterministic finite automaton




EXPSPACE
$\overset{?}{=}$
EXPTIME
$\overset{?}{=}$
PSPACE
$\overset{?}{=}$
NP
$\overset{?}{=}$
P
$\overset{?}{=}$
NL

Class	Resource	Model	Constraint
DTIME(f(n))	Time	DTM	$f(n)$
P	Time	DTM	$O(n^k)$
EXPTIME	Time	DTM	$O(2^{n^k})$
NTIME	Time	NDTM	$f(n)$
NP	Time	NDTM	$O(n^k)$
NEXPTIME	Time	NDTM	$O(2^{n^k})$
DSPACE(f(n))	Space	DTM	$f(n)$
L	Space	DTM	$O(\log(n))$
PSPACE	Space	DTM	$O(n^k)$
EXPSPACE	Space	DTM	$O(2^{n^k})$
NSPACE(f(n))	Space	NDTM	$f(n)$
NL	Space	NDTM	$O(\log(n))$
NPSPACE	Space	NDTM	$O(n^k)$
NEXPSPACE	Space	NDTM	$O(2^{n^k})$

Basic Asymptotic Efficiency Classes



Class	Name	Comments
1	Constant	Algorithm ignores input (i.e., can't even scan input)

Basic Asymptotic Efficiency Classes

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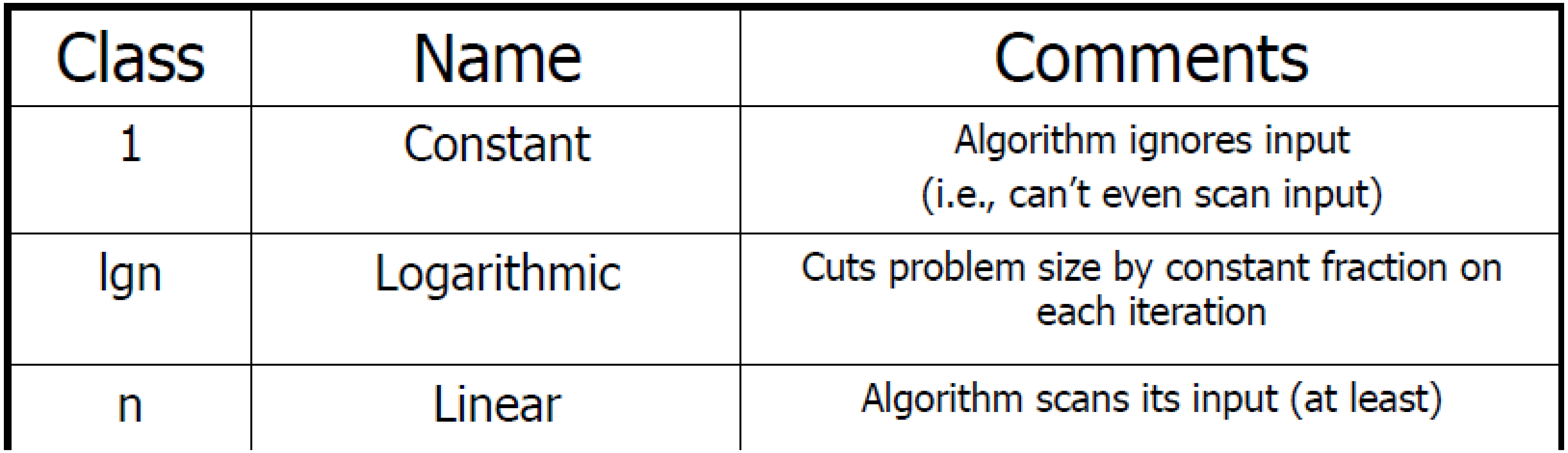
1	Constant	Algorithm ignores input (i.e., can't even scan input)
lgn	Logarithmic	Cuts problem size by constant fraction on each iteration

Basic Asymptotic Efficiency Classes



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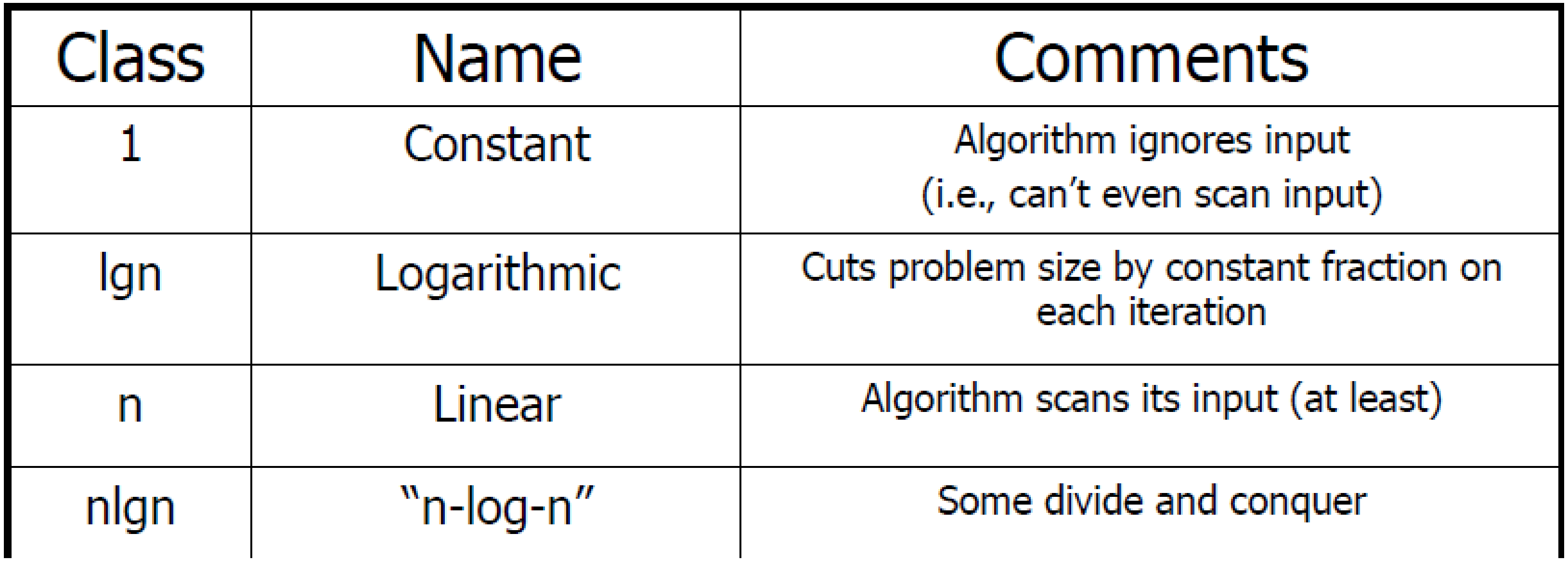
n	Linear	Algorithm scans its input (at least)

Basic Asymptotic Efficiency Classes



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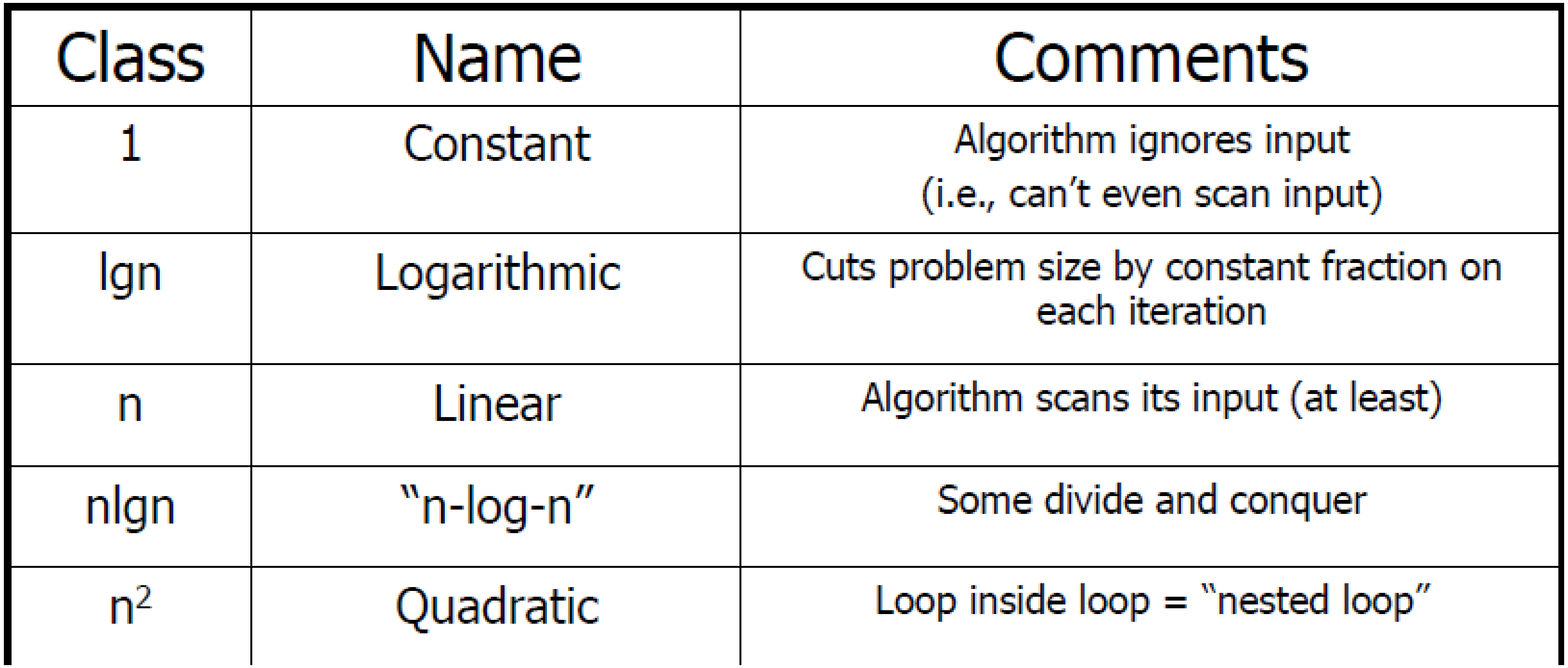
Basic Asymptotic Efficiency Classes



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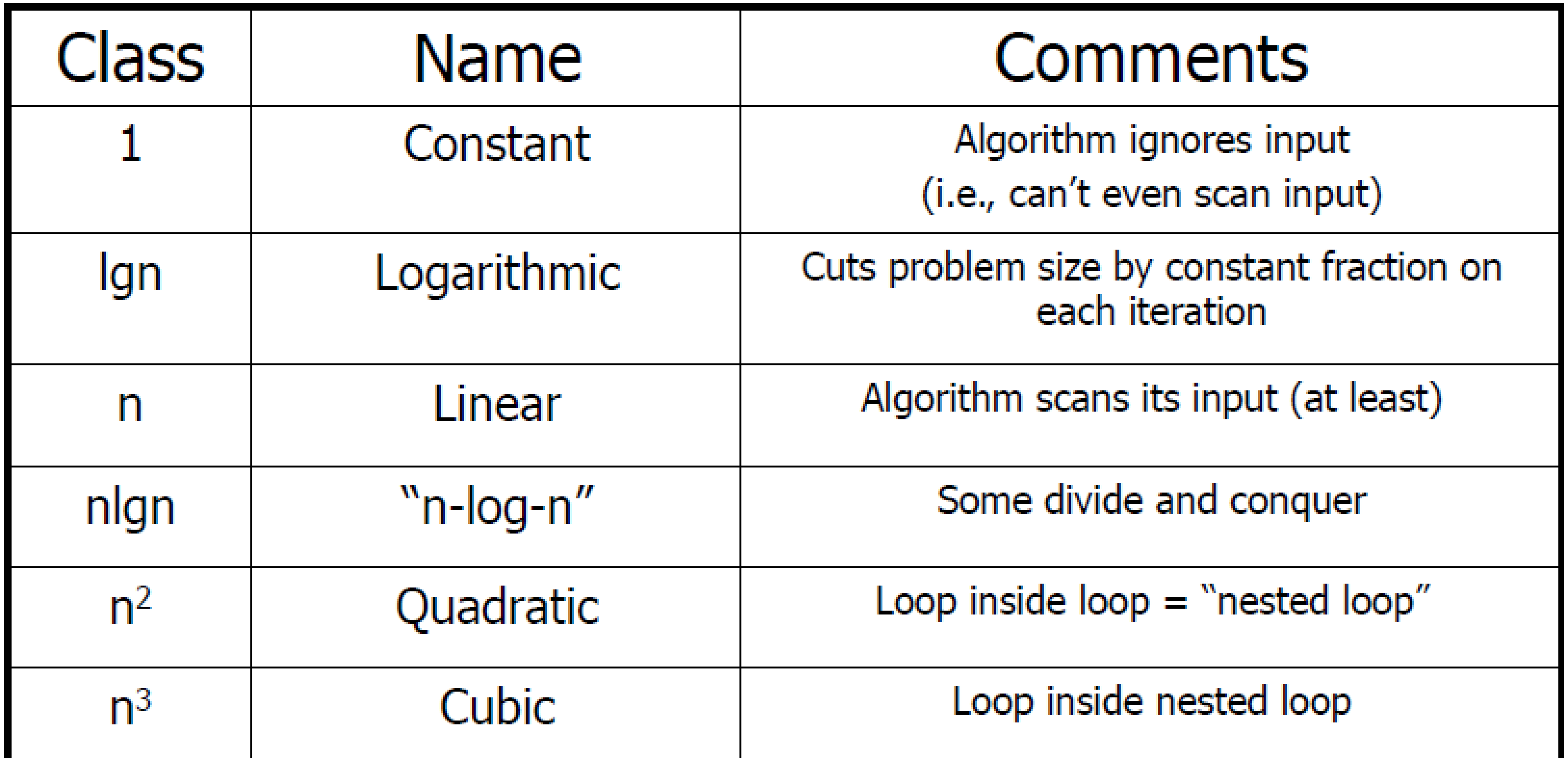
n^2	Quadratic	Loop inside loop = "nested loop"

Basic Asymptotic Efficiency Classes



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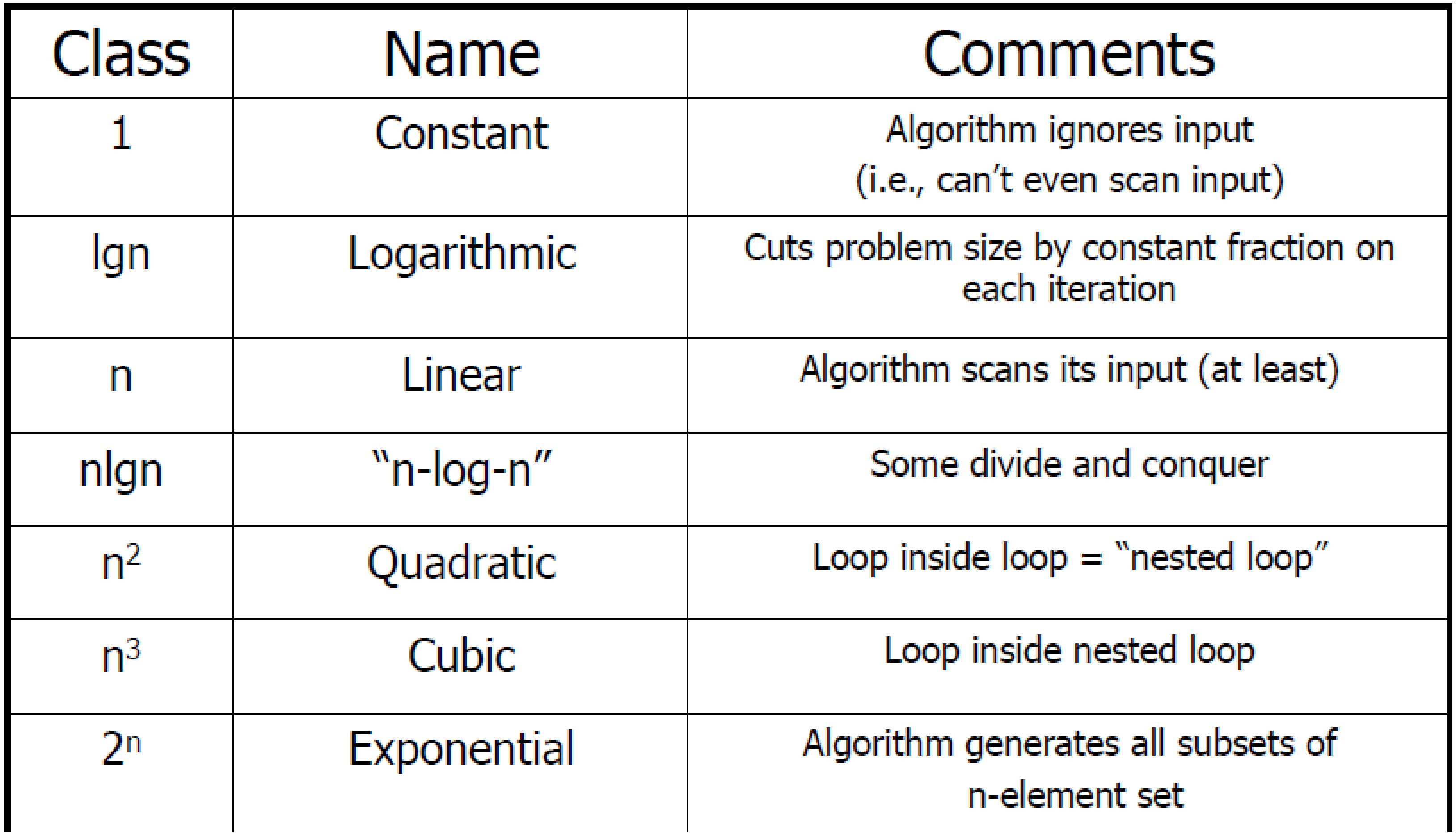
Basic Asymptotic Efficiency Classes



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2^n	Exponential	Algorithm generates all subsets of n-element set

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n^2	Quadratic	Loop inside loop = “nested loop”
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2^n	Exponential	Algorithm generates all subsets of n-element set
n!	Factorial	Algorithm generates all permutations of n-element set

ArrayList vs. LinkedList

	ArrayList	LinkedList
add(element)	O(1)	O(1)
remove(object)	O(n) + O(n)	O(n) + O(1)
get(index)	O(1)	O(n)
set(index, element)	O(1)	O(n) + O(1)
add(index, element)	O(1) + O(n)	O(n) + O(1)
remove(index)	O(n)	O(n) + O(1)
contains(object)	O(n)	O(n)
indexOf(object)	O(n)	O(n)

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