POLITECNICO DI TORINO

e-Lite

Graphs: Cycles

Tecniche di Programmazione – A.A. 2020/2021

Summary

- Definitions
- Algorithms

Definitions

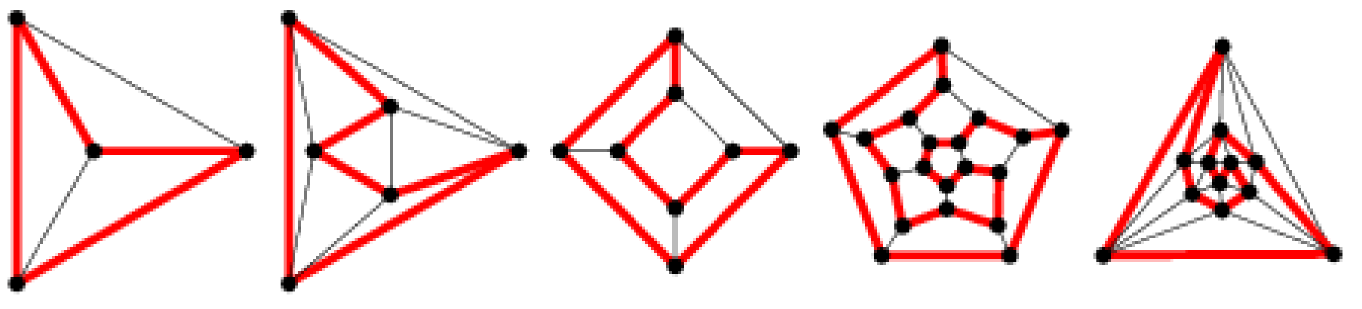
Graphs: Cycles

Cycle

- A **cycle** of a graph, sometimes also called a circuit, is a subset of the edge set of that forms a path such that the first node of the path corresponds to the last.

Hamiltonian cycle

- A cycle that uses each graph vertex of a graph exactly once is called a Hamiltonian cycle.



Hamiltonian path

- A Hamiltonian path, also called a Hamilton path, is a path between two vertices of a graph that visits each vertex exactly once.
  - N.B. does not need to return to the starting point

Eulerian Path and Cycle

- An **Eulerian path**, also called an Euler chain, Euler trail, Euler walk, or "Eulerian" version of any of these variants, is a walk on the graph edges of a graph which **uses each graph edge** in the original graph **exactly once**.
- An **Eulerian cycle**, also called an Eulerian circuit, Euler circuit, Eulerian tour, or Euler tour, is a trail which starts and ends at the **same** graph vertex.

Theorem

- A connected graph has an Eulerian **cycle** if and only if it **all vertices have even degree**.
- A connected graph has an Eulerian **path** if and only if it has **at most two graph vertices of odd degree**.

  - …easy to check!

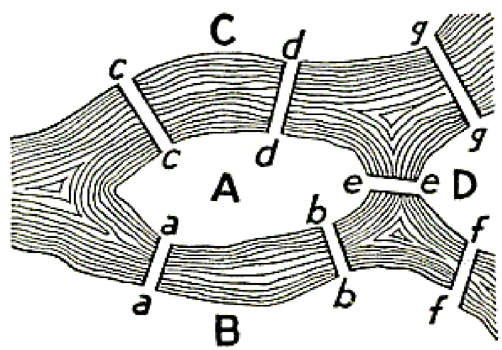
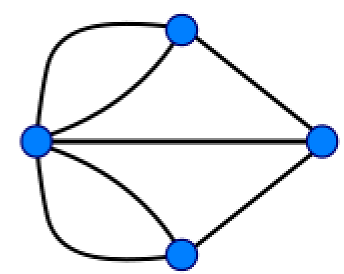


FIGURE 98. *Geographic Map: The Königsberg Bridges.*



Königsberg Bridges

Weighted vs. Unweighted

- Classical versions defined on Unweighted graphs
- Unweighted:
  - Does such a cycle exist?
  - If yes, find at least one
    - Optionally, find all of them
- Weighted
  - Does such a cycle exist?
    - Often, the graph is complete ☺
  - If yes, find at least one
  - If yes, find **the best one** (with **minimum** weight)

Algorithms

Graphs: Cycles

Eulerian cycles: Hierholzer's algorithm (1)

- Choose **any** starting vertex *v*, and **follow a trail** of edges from that vertex until returning to *v*.
  - It is **not** possible to get stuck at any vertex other than *v*, because the even degree of all vertices ensures that, when the trail enters another vertex *w* there must be an unused edge leaving *w*.
  - The tour formed in this way is a **closed** tour, but may **not** cover all the vertices and edges of the initial graph.

Eulerian cycles: Hierholzer's algorithm (2)

- As long as there exists a vertex *v* that belongs to the current tour but that has adjacent edges not part of the tour, **start another trail** from *v*, following **unused** edges until returning to *v*, **and join** the tour formed in this way to the previous tour.

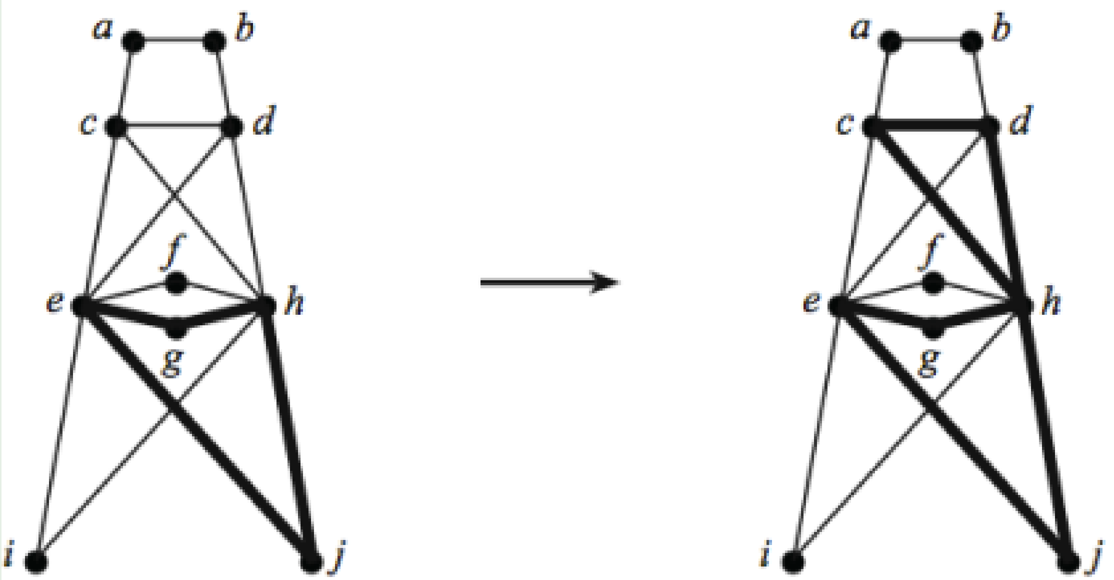
Given: an Eulerian graph G

Find an Eulerian circuit of G.

1. Identify a circuit in G and call it R_1. Mark the edges of R_1. Let $i = 1$.
2. If R_i contains all edges of G, then stop (since R_i is an Eulerian circuit).
3. If R_i does not contain all edges of G, then let v_i be a node on R_i that is incident with an unmarked edge, e_i.
4. Build a circuit, Q_i, starting at node v_i and using edge e_i. Mark the edges of Q_i.
5. Create a new circuit, R_{i+1}, by patching the circuit Q_i into R_i at v_i.
6. Increment i by 1, and go to step (2).

Finding Eulerian circuits
Hierholzer's Algorithm

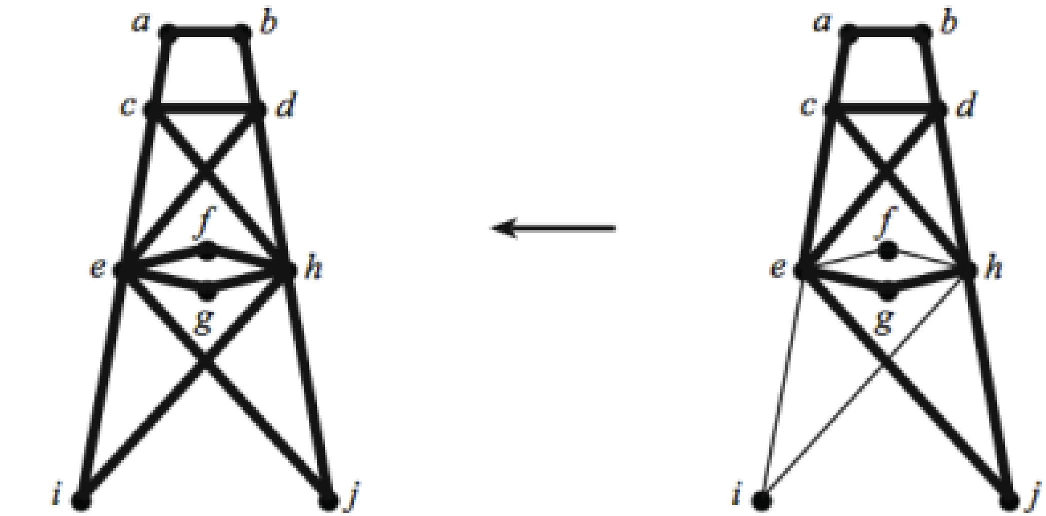
Example



R_1: *e, g, h, j, e*　　　　R_2: *e, g,* **h, d, c, h**, *j, e*

Q_1: *h, d, c, h*　　　　Q_2: *d, b, a, c, e, d*

Example (continued)



R_4: $e, g, \mathbf{h}, \mathbf{f}, \mathbf{e}, \mathbf{i}, \mathbf{h}, d, b, a,$
c, e, d, c, h, j, e

R_3: $e, g, h, \mathbf{d}, \mathbf{b}, \mathbf{a}, \mathbf{c}, \mathbf{e}, \mathbf{d}, c, h, j, e$

Q_3: h, f, e, i, h

Eulerian Circuits in JGraphT

org.jgrapht.alg.cycle

All Classes

Packages

- org.jgrapht
- org.jgrapht.alg
- org.jgrapht.alg.clique
- org.jgrapht.alg.color
- org.jgrapht.alg.connectivity
- org.jgrapht.alg.cycle
- org.jgrapht.alg.decomposition
- org.jgrapht.alg.flow

- HawickJamesSimpleCycles
- HeavyPathDecomposition
- HeavyPathLCAFinder
- HeldKarpTSP
- HelloJGraphT
- HierholzerEulerianCycle
- HopcroftKarpMaximumCardinalityBipartiteMat
- HyperCubeGraphGenerator
- ImmutableDoubleValueGraphAdapter
- ImmutableGraphAdapter
- ImmutableNetworkAdapter
- ImmutableValueGraphAdapter
- ImportException
- *IndependentSetAlgorithm*
- *IndependentSetAlgorithm.IndependentSet*
- IndependentSetAlgorithm.IndependentSetImpl
- IntegerComponentNameProvider
- *IntrusiveEdgesSpecifics*
- IsomorphicGraphMapping
- *IsomorphismInspector*
- IsomorphismUndecidableException
- JGraphXAdapter
- JGraphXAdapterDemo
- JohnsonShortestPaths
- JohnsonSimpleCycles
- KleinbergSmallWorldGraphGenerator
- KolmogorovMinimumWeightPerfectMatching
- KolmogorovMinimumWeightPerfectMatching.D
- KolmogorovMinimumWeightPerfectMatching.St
- KosarajuStrongConnectivityInspector

OVERVIEW PACKAGE **CLASS** USE TREE DEPRECATED INDEX HELP

PREV CLASS NEXT CLASS FRAMES NO FRAMES

SUMMARY: NESTED | FIELD | CONSTR | METHOD DETAIL: FIELD | CONSTR | METHOD

org.jgrapht.alg.cycle

Class HierholzerEulerianCycle<V,E>

java.lang.Object
    org.jgrapht.alg.cycle.HierholzerEulerianCycle<V,E>

Type Parameters:

`V` - the graph vertex type

`E` - the graph edge type

All Implemented Interfaces:

`EulerianCycleAlgorithm<V,E>`

```
public class HierholzerEulerianCycle<V,E>
extends Object
implements EulerianCycleAlgorithm<V,E>
```

An implementation of Hierholzer's algorithm for finding an Eulerian cycle in Eulerian graphs. The algorithm works with directed and undirected graphs which may contain loops and/or multiple (parallel) edges. The running time is linear, i.e. $O(|E|)$ where $|E|$ is the cardinality of the edge set of the graph.

See the Wikipedia article for details and references about Eulerian cycles and a short description of Hierholzer's algorithm for the construction of an Eulerian cycle. The original presentation of the algorithm dates back to 1873 and the following paper: Carl Hierholzer: Über die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren. Mathematische Annalen 6(1), 30–32, 1873.

Author:

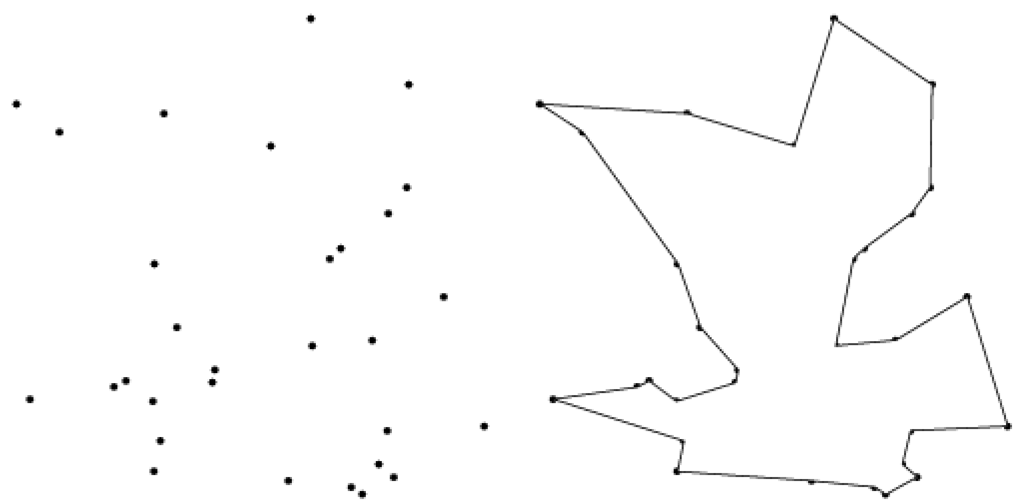
Dimitrios Michail

Constructor Summary

Constructors

Hamiltonian Cycles

- There are theorems to identify **whether** a graph is Hamiltonian (i.e., whether it contains at least one Hamiltonian Cycle)
- **Finding** such a cycle has **no** known efficient solution, in the general case
- Example: the **Traveling Salesman Problem** (TSP)



The Traveling Salesman Problem (TSP)

Weighted or unweighted

Given a collection of cities connected by roads

Find the shortest route that visits each city exactly once.

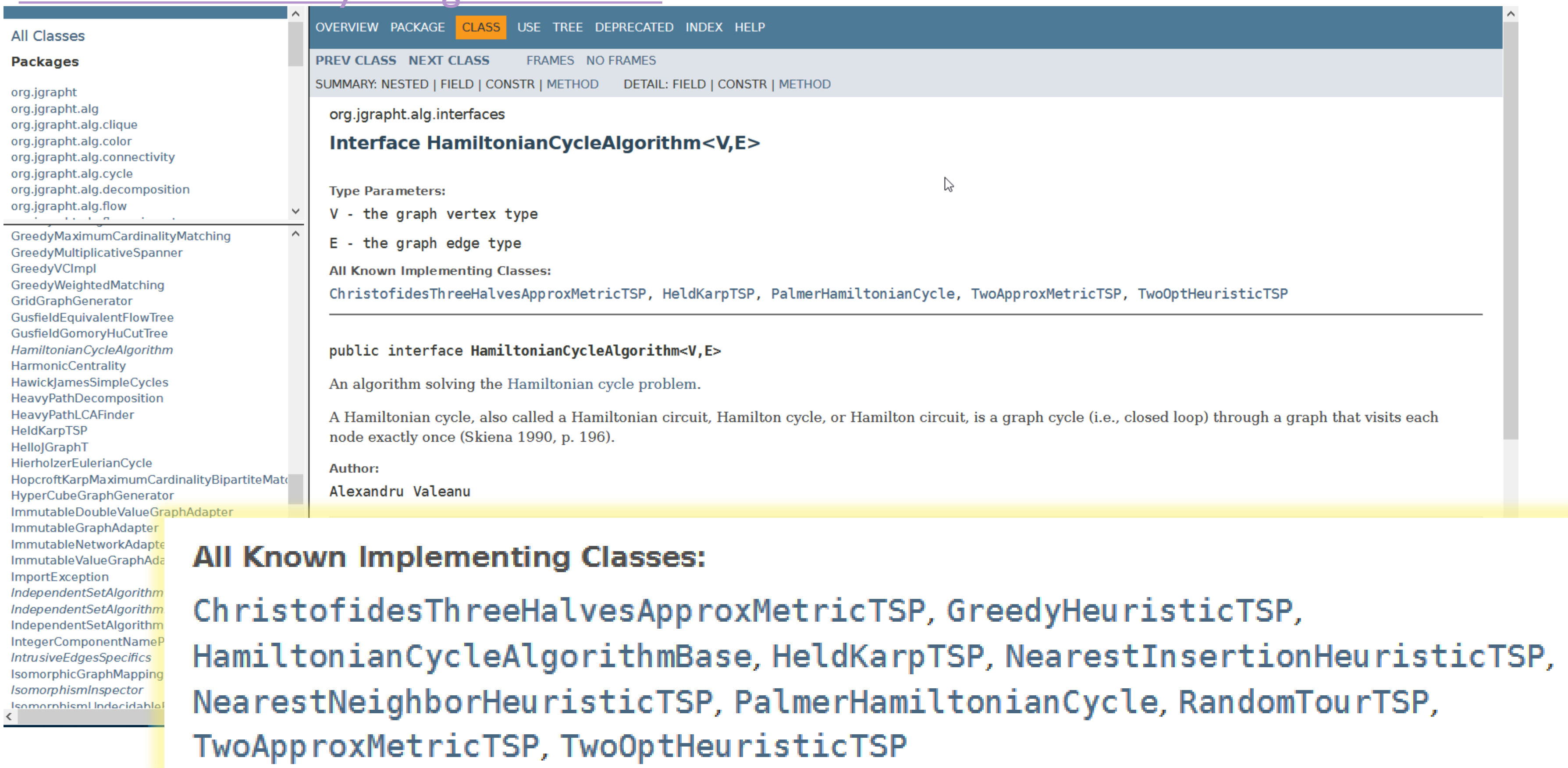
About TSP

- Most notorious NP-complete problem.
- Typically, it is solved with a backtracking algorithm:
  - The best tour found to date is saved.
  - The search backtracks unless the partial solution is cheaper than the cost of the best tour.

Hamiltonian Cycles in JGraphT

https://jgrapht.org/javadoc/org/jgrapht/alg/interfaces/HamiltonianCycleAlgorithm.html

org.jgrapht.alg.**interfaces**



Limitations…

- No exact solution (Approximate algorithms)
  - Class TwoApproxMetricTSP<V,E>
  - Class ChristofidesThreeHalvesApproxMetricTSP<V,E>
  - Class TwoOptHeuristicTSP<V,E>
- Or complete under extra conditions
  - Class PalmerHamiltonianCycle<V,E>
- Or complete but O(2^N)
  - Class HeldKarpTSP<V,E>

ASSUMPTION: G is a metric graph.

1. Compute a minimum weight spanning tree T for G.
2. Perform a depth-first traversal of T starting from any node, and order the nodes of G as they were discovered in this traversal.

   $\Rightarrow$ a tour that is at most twice the optimal tour in G.

Class TwoApproxMetricTSP<V,E>

Resources

- http://mathworld.wolfram.com/
- http://en.wikipedia.org/wiki/Euler_cycle
- Mircea MARIN, Graph Theory and Combinatorics, Lectures 9 and 10, http://web.info.uvt.ro/~mmarin/

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